Surgery in Complex Dynamics.

Carsten Lunde Petersen

IMFUFA at NSM, Roskilde University

Introduction

Magic Lecture 1: Cut and Past surgery using interpolations.

Magic Lecture 2: Quasi-conformal massage, -maps on drugs -.

Magic Lecture 3: Quasi-conformal surgery using real conjugacies.

Magic Lecture 4: Trans-quasi-conformal surgery.

Cubic Blaschke products

For $3 \leq r$ the cubic Blaschke product

$$f(z) = z^2 \frac{z+r}{1+rz}$$

restricts to a real analytic circle homeomorphism (diffeo for 3 < r) with a fixed point at 1 of multiplier (3 - r)/(1 - r). It further has super attracting fixed points at 0 and ∞ of local degree 2.

Fix $r \ge 3$ and let $\theta \in [0,1] \setminus \mathbb{Q}$ be given. Then there exists a (unique) $\eta = \eta(\theta) \in]0,1[$ such that the map

$$f_{\theta}(z) := e^{i2\pi\eta} \cdot f(z)$$

has rotation number θ on the unit circle.

Rotation number

Definition and Theorem 1 (Poincaré). Suppose $g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is an o-p homeomorphism and let $G : \mathbb{R} \longrightarrow \mathbb{R}$ be a lift of $g(e^{i2\pi z})$ to $e^{i2\pi z}$. Then for any $x \in \mathbb{R}$ the limit

$$t = \lim_{n \to \infty} \frac{G^n(x) - x}{n}$$

exists and does not depend on x. Moreover the fractional part $\theta = [t] \in \mathbb{R}/\mathbb{Z}$ is independent of the choice of lift G. It is called the rotation number of g.

Furthermore let $R_{\theta}(z) = e^{i2\pi\theta}z$, and let $w_0 \in \mathbb{S}^1$ be arbitrary. Then there exists a unique o-p semi-conjugacy $h : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ with $h(w_0) = 1$ of g to R_{θ} :

$$h \circ g = R_{\theta} \circ h.$$

Actual conjugacy

Theorem 2. Let $g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be an o-p homeomorphism with irrational rotation number. If g is either: a) (Denjoy) a C^2 diffeomorphism or b) (Yoccoz) a real analytic homeomorphism with at least one critical point. Then the Poincaré semi-conjugacy is a conjugacy.

Note that if r = 3 then condition b) is fulfilled and

if r > 3 then condition a) is fulfilled.

Hence in either case f_{θ} is topologically topologically conjugate to R_{θ} by a unique o-p homeomorphism $h = h_{\theta}$, which is say normalized by fixing h(1) = 1.

Blaschke surgery

Choose a homeomorphic extension $H = H_{\theta} : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ of the circle conjugacy h say with H(0) = 0.

Define a new dynamical system $F_{\theta} = F_{\theta,H} : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$

$$F_{\theta}(z) = \begin{cases} f_{\theta}(z), & z \notin \mathbb{D}, \\ H^{-1} \circ R_{\theta} \circ H(z) & z \in \mathbb{D} \end{cases}$$

Then F_{θ} is a topological degree 2 map which is conformal outside $\overline{\mathbb{D}}$, but it is certainly not conformal on $\overline{\mathbb{D}}$. However if H is quasi conformal, then F is quasi regular and we are in buisness:

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Suppose for the time being that H is q-c. Let σ denote the unique F^* invariant almost complex structure on $\overline{\mathbb{C}}$, satisfying

$$\sigma(z) = \begin{cases} H^*(\sigma_0), & z \in \mathbb{D} \\ \sigma_0, & z \notin \bigcup_{n \ge 0} F^{-n}(D). \end{cases}$$

Let $\phi : (\overline{\mathbb{C}}, \sigma) \longrightarrow (\overline{\mathbb{C}}, \sigma_0)$ be the integrating o-p q-c homeomorphism which is normalized by $\phi(0) = 0$ and $re^{i2\pi\eta}\phi(z)/z \rightarrow 1$ as $z \rightarrow \infty$.

This construction was invented in the case r > 3 by Etienne Ghys in his thesis and ported to the critical case r = 3 by Douady in his 1987 Seminaire Bourbaki.

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Then $P_{\theta}(z) = \phi \circ F_{\theta} \circ \phi^{-1}$ is a quadratic polynomial with fixed point at 0 of multiplier λ_{θ} and with $P_{\theta}(z)/z^2 \to 1$ as $z \to \infty$. That is $P_{\theta}(z) = \lambda_{\theta} z + z^2$.

Moreover P_{θ} has a Siegel disk $\Delta_{\theta} = \phi(\mathbb{D})$ with quasi-circle boundary $\phi(\mathbb{S}^1)$.

Finally if r = 3 so that 1 is a critical point, then the quasi-circle boundary of Δ_{θ} contains the critical point $-\lambda_{\theta}/2$ and if r > 3 then it does not.

Quasi-conformal extension?

Recall that

Definition 3. A homeomorphism $F : \Gamma \longrightarrow \gamma$, where $\Gamma, \gamma \subset \mathbb{C}$ is called quasi-symmetric if there exists a constant C > 1 such that for any triplet of points $x, y, z \in \Gamma$ with |x - y| = |y - z|:

$$1/C \le \frac{|F(x) - F(y)|}{|F(y) - F(z)|} \le C.$$

A classical result due to Ahlfors and Beurling is that **Theorem 4.** A homeomorphism $h : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ has a quasi conformal extension $H : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$ if and only if h is quasi-symmetric.

So the exiting question is if h_{θ} is q-s?

Conditions for quasi-symmetri

Michel Herman (who was supervisor for Etienne Ghys) proved first that there exists r > 3 and irrational rotation numbers θ such that f_{θ} is q-s conjugate to R_{θ} , but not analytically conjugate.

Ghys used this and his construction to prove that there are quadratic polynomials with a Siegel disk, whose boundary is a quasi-circle not containing the critical point.

This was/is interesting, because Douady previously had proved that. If P_{θ} has a Siegel disk Δ_{θ} and if the critical point $-\lambda_{\theta}/2$ does not belong to $\overline{\Delta_{\theta}}$. Then the Julia set J_{θ} is not locally connected.

Herman-Swiatek

Herman who attended the Bourbaki seminar of Douady proved within a few days after the seminar the following theorem, using ideas of Grazyk Swiatek:

Theorem 5 (Herman-Swiatek). Let $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be areal analytic, o-p circle homeomorphism with a critical point. Then the Poincaré-Yoccoz conjugacy is quasi-symmetric if and only if θ is of bounded type.

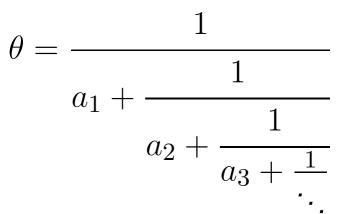
Thus we may conclude:

Theorem 6. For θ of bounded type the quadratic polynomial $P_{\theta}(z) = \lambda_{\theta} z + z^2$ has a Siegel disk, whose boundary is a quasi-circle containing the critical point.

Remark the converse is also true, but the proof is more elaborate.

Bounded type

Definition 7. We may write any irrational number $\theta \in [0, 1]$ in a unique way as a continued fraction with positive integer coefficients (called partial fractions):



The number θ is said to be of bounded type if the sequence (a_n) is bounded.

The prime example is the golden mean with

$$a_n \equiv 1.$$

Similar applications

Theorem 8. (S. Zakeri) For θ of bounded type any cubic polynomial $P_{\theta,a}(z) = \lambda_{\theta} z + az^2 + z^3$ has a Siegel disk $\Delta_{\theta,a}$ whose boundary contains at least one and in particular cases both the finite critical points of $P_{\theta,a}$.

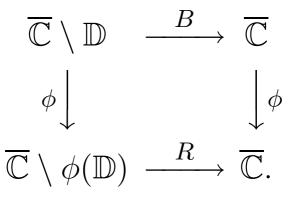
Theorem 9. (*M.* Yampolsky and S. Zakeri) For $0 < \theta, \tau < 1$ of bounded type with $\theta + \tau \neq 1$ the quadratic rational map

$$R_{\theta,\tau}(z) = z \frac{z + \lambda_{\theta}}{1 + \lambda_{\tau} z}$$

has a Siegeld disks $\Delta_{\theta,\tau}^0$ and $\Delta_{\theta,\tau}^\infty$ around 0 and ∞ respectively, with disjoint quasi circle boundaries each of which contains a critical point.

Similar applications II

Theorem 10. (L. Ma) Suppose the Blaschke product B restricts to an o-p degree $d \ge 2$ covering $B : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ and that a) every critical point $c \in \mathbb{S}^1$ for B is not recurrent to a critical point, b) every periodic point for B on \mathbb{S}^1 is repelling. Then there exists a rational map R and an o-p q-c homeomorphism $\phi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ such that



and $\Lambda = \phi(\mathbb{D})$ is a super attracting basin for R on which R is conformally conjugate to z^d .

More refined similar applications

Theorem 11. (A. Cheritat) For any θ of bounded type the parabolic quadratic polynomial $Q_{\frac{1}{4}}(z) = z^2 + \frac{1}{4}$ has a virtual-Siegel disk Δ with rotation number θ and quasi-circle boundary containing the critical point.

Theorem 12. (*G. Zhang*) For any θ of bounded type the map $\lambda_{\theta} \sin z$ has a Siegel disk Δ_{θ} with quasi-circle boundary containing the two nearest critical points $\pm \pi/2$, but none of the others.