Filtered and Intersection Homology

Jon Woolf, work in progress with Ryan Wissett

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Part I

Review of intersection homology

Singular intersection homology

Perversities

A perversity on a topologically stratified space X is a function $p\colon \{\text{strata of } X\} \to \mathbb{Z}$. If

1. $p(S) = p(\operatorname{codim} S)$ for some $p\colon \mathbb{N} \to \mathbb{Z}$
2. $p(k) = 0$ for $k \leq 2$
3. $p(k+1) = p(k)$ or $p(k) + 1$.

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Examples

- the zero perversity $0(k) = 0$
- the top perversity $t(k) = \max\{k-2, 0\}$
- the lower middle perversity $m(k) = \max\{\lfloor (k-2)/2 \rfloor, 0\}$
- the upper middle perversity $n(k) = \max\{\lceil (k-2)/2 \rceil, 0\}$

GM perversities p and q are complementary if $p + q = t$.

Intersection homology and Poincaré duality

Intersection homology

A perversity picks out a subcomplex of intersection chains in S_*X:

$$\Delta^i \xrightarrow{\sigma} X \ p\text{-allowable} \iff \sigma^{-1}S \subset (i - \operatorname{codim} S + p(S))\text{-skeleton}$$
$$c \in S_iX \ p\text{-allowable} \iff \text{all simplices in } c \text{ are } p\text{-allowable}$$

Let $I^pS_*X = \{c \mid c, \partial c \text{ are } p\text{-allowable}\}$ and I^pH_*X its homology.

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Theorem (Goresky–MacPherson '80)

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Theorem (Goresky–MacPherson '80)

X compact, oriented n-dim pseudomfld, p, q complementary GM perversities $\implies$ $\exists$ intersection pairing

$$I^pH_iX \times I^qH_{n-i}X \to \mathbb{Z}$$

which is non-degenerate over $\mathbb{Q}$.

Part II

Filtered homology

Filtered spaces and depth functions

Filtered spaces

A filtered space X_α is a topological space with a filtration

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_\infty = X.$$

A filtered map $f\colon X_\alpha \to Y_\beta$ is a map with $f(X_k) \subset Y_k \ \forall\ k \in \mathbb{N}$.

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Depth functions

The filtration on X_α is encoded in the depth function $\alpha\colon X \to \mathbb{N}_\infty$ where

$$\alpha(x) = k \iff x \in X_k - X_{k-1}$$

so $X_k = \alpha^{-1}\{0, \ldots, k\}$ and $f\colon X_\alpha \to Y_\beta$ filtered $\iff \alpha \geq \beta \circ f$.

Examples of filtered spaces

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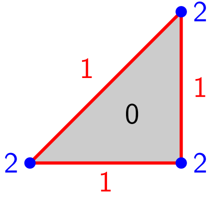
1. A filtered space of depth ≤ 1 is a pair $X_0 \subset X_1 = X$; a filtered map of such is a map of pairs.
2. Filtering a CW complex by its skeleta fully faithfully embeds CW complexes and cellular maps into filtered spaces.
3. Let Δ^n_δ be the standard simplex filtered by depth function $\delta(t_0, \ldots, t_n) = \#\{i \mid t_i = 0\}$, e.g.



The face maps $\Delta^{i-1}_{\delta+1} \hookrightarrow \Delta^i_\delta$ are filtered.

Filtered homology

For filtered X_α define $S_i X_\alpha = \mathbb{Z}\{\Delta^i_\delta \to X_\alpha\}$. Note

$$\partial\colon S_i X_\alpha \to S_{i-1} X_{\alpha-1}$$

where $(\alpha - 1)(x) = \max\{\alpha(x) - 1, 0\}$.

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Definition

The filtered i-chains on X_α are

$$FS_i X_\alpha = \{c \in S_i X_\alpha \mid \partial c \in S_{i-1} X_\alpha\}.$$

The filtered homology $FH_* X_\alpha$ is the homology of $FS_* X_\alpha$.

Properties of filtered homology

Functoriality

Filtered $f\colon X_\alpha \to Y_\beta$ induces a chain map $FS_*X_\alpha \to FS_*Y_\beta$ and

$$f_*\colon FH_*X_\alpha \to FH_*Y_\beta.$$

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Relative long exact sequence

For filtered $f\colon X_\alpha \to Y_\beta$ where the underlying map is an inclusion we define $FH_i(Y_\beta, X_\alpha) = H_i\left(FS_*Y_\beta / FS_*X_\alpha\right)$. There is a LES

$$\cdots \to FH_*X_\alpha \to FH_*Y_\beta \to FH_*(Y_\beta, X_\alpha) \to FH_{*-1}X_\alpha \to \cdots$$

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Excision

For $Z_\alpha \subset Y_\alpha \subset X_\alpha$ with $\overline{Z} \subset Y^o$ there are isomorphisms

$$FH_*(X_\alpha - Z_\alpha, Y_\alpha - Z_\alpha) \cong FH_*(X_\alpha, Y_\alpha).$$

Simple examples of filtered homology

Cones

For $[x, t] \in CX$, the cone on X, and $d > 1$ have

$$\beta[x,t] = \begin{cases} \alpha(x) & t > 0 \\ d & t = 0 \end{cases} \implies FH_i CX_\beta \cong \begin{cases} FH_i X_\alpha & i < d-1 \\ 0 & i \geq d-1. \end{cases}$$

When $d \leq 1$ obtain homology of a point.

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Suspended torus

Let $X_\alpha = \Sigma T^2$ where $\alpha(x) = 2$ at suspension points and 0 elsewhere. Then

$$FH_i X_\alpha = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}^2 & i = 2 \\ \mathbb{Z} & i = 3. \end{cases}$$

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Perversities and filtrations

Given stratified X and perversity p define a depth function

$$\hat{p}(x) = \operatorname{codim} S - p(S)$$

for $x \in S$. The identity $X_{\hat{p}} \to X_{\hat{q}}$ is filtered $\iff p \leq q$.

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- $X_{\hat{0}} = \left(X^0 \subset X^1 \subset \cdots \subset X^k \subset \cdots \subset X\right)$

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- p is a Goresky–Macpherson perversity $\iff X_{\hat{p}}$ is filtration by those X^k with $p(k) = p(k+1)$
- Complementary perversities p and q give 'complementary' filtrations: X^k with $k \geq 2$ appears in either $X_{\hat{p}}$ or $X_{\hat{q}}$.

Intersection homology is filtered homology

An elementary calculation gives

$$\begin{aligned}\Delta^i_\delta \xrightarrow{\sigma} X_{\hat{p}} \text{ filtered} &\iff \sigma^{-1}S \subset (i - \operatorname{codim} S + p(S))\text{-skeleton} \\ &\iff \sigma \ p\text{-allowable}\end{aligned}$$

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- Functoriality of $FH_* \implies$ known functoriality of IH_*
- Intersection homology is a filtered homotopy invariant
- Filtered homology LES gives relative LES for IH_*, and obstruction sequence for change of perversities.

Part III

Spectral sequence of a filtered space

The spectral sequence

For filtered X_α the singular complex S_*X has natural filtration

$$0 \hookrightarrow S_*X_\alpha \hookrightarrow S_*X_{\alpha-1} \hookrightarrow \cdots \hookrightarrow S_*X$$

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yielding a spectral sequence with E^0-page

$$\begin{array}{ccccc}
S_0X_\alpha & & S_1X_\alpha & & S_2X_\alpha \\
\downarrow & & \downarrow & & \downarrow \\
0 & & \frac{S_0X_{\alpha-1}}{S_0X_\alpha} & & \frac{S_1X_{\alpha-1}}{S_1X_\alpha} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & \frac{S_0X_{\alpha-2}}{S_0X_{\alpha-1}}
\end{array}$$

converging to $Gr_\bullet H_*X$ where

$$Gr_iH_jX = \frac{\{[c] \in H_jX \mid c \in S_jX_{\alpha-i}\}}{\{[c] \in H_jX \mid c \in S_jX_{\alpha-i+1}\}}.$$

The spectral sequence

The singular complex S_*X of filtered X_α has natural filtration

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yielding a spectral sequence with E^1-page

$$\begin{array}{ccccc} FS_0X_\alpha & \longleftarrow & FS_1X_\alpha & \longleftarrow & FS_2X_\alpha \\ \\ 0 & \longleftarrow & \frac{FS_0X_{\alpha-1}}{S_0X_\alpha + \partial S_1X_\alpha} & \longleftarrow & \frac{FS_1X_{\alpha-1}}{S_1X_\alpha + \partial S_2X_\alpha} \\ \\ 0 & \longleftarrow & 0 & \longleftarrow & \frac{FS_0X_{\alpha-2}}{S_0X_{\alpha-1} + \partial S_1X_{\alpha-1}} \end{array}$$

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The spectral sequence

The singular complex S_*X of filtered X_α has natural filtration

$$0 \hookrightarrow S_*X_\alpha \hookrightarrow S_*X_{\alpha-1} \hookrightarrow \cdots \hookrightarrow S_*X$$

yielding a spectral sequence with E^2-page

$$\begin{array}{ccc}
FH_0X_\alpha & FH_1X_\alpha & FH_2X_\alpha \\
0 & FH_0(X_{\alpha-1}, X_\alpha) & FH_1(X_{\alpha-1}, X_\alpha) \\
0 & 0 & FH_0(X_{\alpha-2}, X_{\alpha-1})
\end{array}$$

converging to $Gr_\bullet H_*X$ where

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The spectral sequence

The singular complex S_*X of filtered X_α has natural filtration

$$0 \hookrightarrow S_*X_\alpha \hookrightarrow S_*X_{\alpha-1} \hookrightarrow \cdots \hookrightarrow S_*X$$

yielding a spectral sequence with E^∞-page

$$\begin{matrix} Gr_0 H_0 X & Gr_0 H_1 X & Gr_0 H_2 X \\ 0 & Gr_1 H_0 X & Gr_1 H_1 X \\ 0 & 0 & Gr_2 H_0 X \end{matrix}$$

converging to $Gr_\bullet H_* X$ where

$$Gr_i H_j X = \frac{\{[c] \in H_j X \mid c \in S_j X_{\alpha-i}\}}{\{[c] \in H_j X \mid c \in S_j X_{\alpha-i-1}\}}.$$

Examples of the spectral sequence

X_α CW-complex with skeletal filtration

E^2-page is cellular chain complex:

$$
\begin{array}{ccc}
FH_0X_\alpha & 0 & 0 \\
0 & 0 & FH_1(X_{\alpha-1}, X_\alpha) \\
0 & 0 & 0
\end{array}
$$

Examples of the spectral sequence

X_α CW-complex with skeletal filtration

$E^3 = E^\infty$-page is cellular homology:

$$\begin{matrix} H_0^{\text{cell}}X & 0 & 0 \\ 0 & 0 & H_1^{\text{cell}}X \\ 0 & 0 & 0 \end{matrix}$$

Examples of the spectral sequence

$X_\alpha = \Sigma T^2$ with α(suspension points) $= 3$

E^2-page:

$\mathbb{Z}$	$\mathbb{Z}^2$	0	$\mathbb{Z}$
0	0	0	$\mathbb{Z}^4$
0	0	0	0

Examples of the spectral sequence

$X_\alpha = \Sigma T^2$ with α(suspension points) $= 3$

$E^3 = E^\infty$-page:

$\mathbb{Z}$	0	0	$\mathbb{Z}$
0	0	0	$\mathbb{Z}^2$
0	0	0	0

Part IV

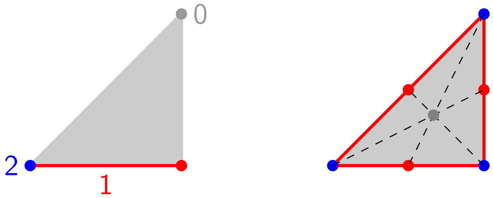
Cap products and Poincaré Duality?

Alternative filtration for simplices

Let $\Delta^n_{\delta'}$ denote the n-simplex with filtration

$$\delta'(t_0, \ldots, t_n) = \min\{i \mid t_{n-i} \neq 0\}$$

and $FS'_* X_\alpha$ the associated complex of filtered chains.



Proposition

There is a homotopy equivalence $FS_* X_\alpha \simeq FS'_* X_\alpha$ provided by composition with $id\colon \Delta^n_\delta \to \Delta^n_{\delta'}$ and barycentric subdivision. So filtered homology can be computed using either complex.

Cap products

Filtered homology as a module

The inclusions of the 'back' faces of $\Delta^n_{\delta'}$ are filtered. The usual cap product formula restricts to $S^iX \otimes S'_jX_\alpha \to S'_{j-i}X_\alpha$ inducing

$$H^iX \otimes FH_jX_\alpha \to FH_{j-i}X_\alpha,$$

so that FH_*X_α is an H^*X-module.

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Generalised Poincaré duality?

A more refined approach should yield a cap product

$$FH^iX_{\hat{p}} \otimes FH_jX_{\hat{0}} \to FH_{j-i}X_{\hat{q}-1}$$

where $p + q = t$.

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so that $FH_* X_\alpha$ is an $H^* X$-module.

Generalised Poincaré duality?

A more refined approach should yield a cap product

$$FH^i X_{\hat{p}} \otimes FH_j X_{\hat{0}} \to FH_{j-i} X_{\hat{q}-1}$$

where $p + q = t$. If we can improve this to

$$FH^i X_{\hat{p}} \otimes FH_j X_{\hat{0}} \to FH_{j-i} X_{\hat{q}}$$

then generalised Poincaré duality would follow.