Intersection cohomology and perverse sheaves

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Notation and conventions

- X complex projective variety, singular set Σ
- X embedded in non-singular projective M
- consider sheaves of $\mathbb{C}$-vector spaces in analytic topology
- $D\mathrm{Sh}_c(X)$ algebraically constructible derived category
- write f_* etc not Rf_* (all functors will be derived)
- a 'local system' on a stratum S is placed in degree $-\dim S$
- Poincaré–Verdier duality is an equivalence

$$D = D\mathrm{Sh}_c(X)^{\mathrm{op}} \to D\mathrm{Sh}_c(X)$$

Many of the results, as well as generalisations to other settings, can be found in [Dim04, Sch03, KS90, GM88, dCM09].

Part I

Perverse sheaves

Poincaré duality

When X non-singular and $\mathcal{L} \cong D\mathcal{L}$ is a self-dual local system

$$\begin{aligned} H^i(X;\mathcal{L}) &= H^i(p_*\mathcal{L}) \qquad \text{where } p : X \to \mathrm{pt} \\ &\cong H^i(p_*D\mathcal{L}) \\ &\cong H^i(Dp_*\mathcal{L}) \\ &\cong DH^{-i}(p_*\mathcal{L}) \\ &\cong DH^{-i}(X;\mathcal{L}) \end{aligned}$$

so that we have Poincaré duality. When X singular $D\mathcal{L}$ not in general a local system so. . .

Poincaré duality for singular spaces

Two possible approaches to extending duality to singular spaces:

$$(X; \mathcal{L})$$

$X - \Sigma \xrightarrow{\jmath} X$

$IX \xrightarrow{f} X$

$(X; \mathrm{IC}_X(\jmath^* \mathcal{L}))$

$(IX; f^* \mathcal{L})$

Intersection cohomology
$IH^i(X) = H^i(X; \mathrm{IC}_X(\jmath^* \mathcal{L}))$
[GM80, GM83a]

Intersection spaces
$H^i(IX; f^* \mathcal{L})$
[Ban10]

Intermediate extensions and intersection cohomology

A self-dual local system $\mathcal{L}$ on a stratum $\jmath_S : S \hookrightarrow X$ has two (dual) extensions, connected by a natural morphism

$$\jmath_{S!}\mathcal{L} \to \jmath_{S*}\mathcal{L}.$$

Theorem ([BBD82])

There is a t-structure on $DSh_c(X)$ preserved by the duality D

$$Perv(X) \underset{{}^pH^0}{\overset{}{\rightleftarrows}} DSh_c(X)$$

The intermediate extension $\jmath_{S!*}\mathcal{L} = \mathrm{IC}_{\overline{S}}(\mathcal{L})$ is the image

$${}^pH^0(\jmath_{S!}\mathcal{L}) \twoheadrightarrow \jmath_{S!*}\mathcal{L} \hookrightarrow {}^pH^0(\jmath_{S*}\mathcal{L})$$

It exists for any $\mathcal{L}$, and is self-dual whenever $\mathcal{L}$ is so.

Perverse sheaves [BBD82]

For a Whitney stratification $\mathbb{S}$ of X by complex varieties we say $\mathcal{E}$ is perverse $\iff$ $\mathcal{E}$ is $\mathbb{S}$-constructible and

$$\begin{cases} H^i(\jmath_S^! \mathcal{E})_x = 0 & \text{for } i < -\dim S \\ H^i(\jmath_S^* \mathcal{E})_x = 0 & \text{for } i > -\dim S \end{cases}$$

for all x in each S. If $\mathcal{E}$ perverse for one stratification then it is perverse for any stratification for which it is constructible. Let $\mathsf{Perv}(X) = \operatorname{colim}_{\mathbb{S}} \mathsf{Perv}_{\mathbb{S}}(X)$.

Examples

- local system on a closed stratum S
- intermediate extensions.

$\mathsf{Perv}_{\mathbb{S}}(X)$ is glued from the categories of local systems (with our shift!) on the strata, each of which is preserved by duality.

Properties of perverse sheaves

It is traditional to remark that perverse sheaves are neither sheaves nor perverse. But they do have nice algebraic properties

- Perv(X) is a stack
- Perv(X) has finite length
- the simple objects are the $\jmath_{S!*}\mathcal{L}$ for S and $\mathcal{L}$ irreducible

Theorem ([BBD82, Sai88, Sai90, dCM05])

The pushforward under a proper map of a simple perverse sheaf[1] *is a direct sum of shifted simple perverse sheaves*[1].

This algebraic result has many important consequences. For instance, it implies that $H^*(\widetilde{X}) \cong IH^*(X) \oplus A^*$ for any resolution $\widetilde{X} \to X$. Combining it with Hodge theory yields the Hard Lefschetz Theorem for $IH^*(X)$.

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[1] of geometric origin

Part II

Links with Morse theory

Stratified Morse theory [GM83b]

Fix stratification $\mathbb{S}$ of $X \subset M$. Say $x \in S$ is critical for smooth $f : M \to \mathbb{R}$ if it is critical for $f|_S$. Then f is Morse if

- critical values distinct
- each critical point in S is non-degenerate for $f|_S$
- $d_x f$ is non-degenerate at each critical point x.

Definition

The normal Morse data for $\mathcal{E}$ at critical $x \in S$ is

$$\mathsf{NMD}(\mathcal{E}, f, x) = R\Gamma_{\{f \geq f(x)\}}(\mathcal{E}|_{N \cap X})_x$$

where N is a complex analytic normal slice to S in M. Depends only on $\mathcal{E}$ and stratum $S \ni x$, so we write $\mathsf{NMD}(\mathcal{E}, S)$.

Examples

$\operatorname{codim} S = 0 \Rightarrow \mathsf{NMD}(\mathcal{E}, S) \cong \mathcal{E}_x$. X non-singular, $\mathcal{L}$ local system and $\operatorname{codim} S > 0 \Rightarrow \mathsf{NMD}(\mathcal{L}, S) = 0$.

Purity is perverse

Definition

$\mathcal{E}$ is pure if NMD $(\mathcal{E}, S)$ concentrated in degree $-\dim S$.

If $x \in S$ is critical for Morse f and $\mathcal{E}$ is pure then

$$H^i(X_{\leq fx-\epsilon}, X_{\leq fx+\epsilon}; \mathcal{E}) \cong \begin{cases} \text{NMD}\,(\mathcal{E}, S) & i = \lambda - \dim S \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda =$ index at x of $f|_S$. For pure $\mathcal{E}$, critical points in S 'contribute' in degrees from $-\dim S$ to $\dim S$. Hence

$$H^i\,(X; \mathcal{E}) = 0 \text{ for } |i| > \dim X.$$

Theorem ([KS90])

$\mathcal{E}$ is perverse $\iff$ $\mathcal{E}$ is pure

Example: intersection cohomology of curves

When X curve and $\mathcal{E} = \mathrm{IC}_X(\mathbb{C})$

$$\mathrm{NMD}\,(\mathcal{E}, x) = \begin{cases} \mathbb{C}^{m_x - b_x} & x \text{ singular} \\ \mathbb{C} & x \text{ non-singular.} \end{cases}$$

Note that the 'Morse group' may not be one-dimensional, e.g. for a higher order cusp, and also that it may vanish, e.g. for a node:

This corresponds to the fact that intersection cohomology is invariant under normalisation.

Lefschetz hyperplane type theorems

If $S \subset \mathbb{C}^n$ then any Morse critical point for a distance function $f|_S$ has index $\leq \dim S$. Therefore for affine $\jmath : U \hookrightarrow X$ and perverse $\mathcal{E}$

$$H^i(U; \mathcal{E}|_U) = 0 \quad \text{for } i > 0.$$

In particular $IH^i(U) = 0$ for $i > 0$.

Theorem ([GM83b])

If H is a generic hyperplane in $\mathbb{CP}^m$ then $IH^i(X) \to IH^i(X \cap H)$ is an isomorphism for $i < -1$ and injective when $i = -1$.

Theorem ([BBD82])

The extensions $\jmath_!$ and $\jmath_*$ preserve perverse sheaves. In particular if U is a stratum with local system $\mathcal{L}$ then $\jmath_!\mathcal{L}$ and $\jmath_*\mathcal{L}$ are perverse.

Part III

Links with symplectic geometry

Characteristic cycles

Fix stratification $\mathbb{S}$. The characteristic cycle [BDK81] of $\mathcal{E}$ is

$$\mathrm{CC}(\mathcal{E}) = \sum_S (-1)^{\dim S} \chi(\mathrm{NMD}(\mathcal{E}, S)) \overline{T_S^* M}$$

where $T_S^* M$ is the conormal bundle to S in M. When $\mathcal{E}$ perverse $\mathrm{CC}(\mathcal{E})$ is effective. The characteristic cycle is independent of $\mathbb{S}$.

Examples

- If $\mathcal{L}$ local system on closed S then $\mathrm{CC}(\mathcal{L}) = \mathrm{rank}(\mathcal{L})\, T_S^* M$.
- If X is a curve then

$$\mathrm{CC}(\mathrm{IC}_X(\mathbb{C})) = \overline{T_{X-\Sigma}^* M} + \sum_{x \in \Sigma} (m_x - b_x) T_x^* M$$

and $\mathrm{CC}(\mathbb{C}_X) = \overline{T_{X-\Sigma}^* M} + \sum_{x \in \Sigma} (1 - m_x) T_x^* M$.

Properties of characteristic cycles

Theorem ([BDK81])

The Brylinski–Dubson–Kashiwara index formula states that

$$\chi(X;\mathcal{E}) = CC(\mathcal{E}) \cdot T^*_M M.$$

where the dot denotes intersection in T^*M.

Example

If X a curve then $\chi(X;\mathrm{IC}_X(\mathbb{C})) = -\chi(X) + \sum_{x\in\Sigma}(1-b_x)$.

Theorem ([KS90])

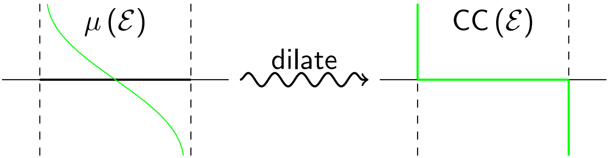
- $CC(\mathcal{E})$ *depends only on* $[\mathcal{E}] \in K(DSh_c(X))$
- $CC(D\mathcal{E}) = CC(\mathcal{E})$
- f *proper* $\Rightarrow CC(f_*\mathcal{E}) = f_* CC(\mathcal{E})$
- f *transversal* $\Rightarrow CC(f^*\mathcal{E}) = f^* CC(\mathcal{E})$.

Nadler and Zaslow's categorification [NZ09]

M real-analytic manifold, $D\mathrm{Sh}_c(M)$ real-an. constr. der. category

$$\begin{array}{ccc} D\mathrm{Sh}_c(M) & \xrightarrow[\mu]{\simeq} & D\mathrm{Fuk}(T^*M) \\ \big\downarrow & & \big\downarrow \text{ dilate} \\ K\left(D\mathrm{Sh}_c(M)\right) & \xrightarrow[\mathrm{CC}]{\simeq} & \mathrm{L}_{\mathrm{con}}(T^*M) \end{array}$$

The micro-localisation μ sends a 'standard open' $\jmath_*\mathbb{C}_U$ to $\Gamma d \log m$ where $m|_U > 0$ and $m|_{\partial U} = 0$. E.g. when $M = \mathbb{R}$ and $\mathcal{E} = \jmath_*\mathbb{C}_{(0,1)}$



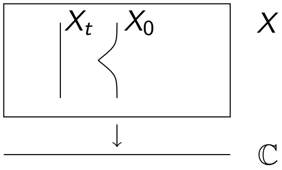
Part IV

Links with representation theory

Nearby and vanishing cycles ...

Let $h : X \to \mathbb{C}$ be regular and $X_t = h^{-1}(t)$. There is a triangle

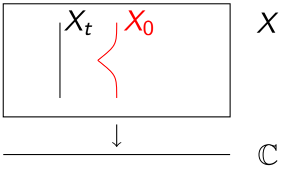
$$\mathcal{E}|_{\mathrm{Re}(h)<0}[-1] \to R\Gamma_{\mathrm{Re}(h)\geq 0}(\mathcal{E}) \to \mathcal{E} \to \mathcal{E}|_{\mathrm{Re}(h)<0}.$$



Nearby and vanishing cycles ...

Let $h : X \to \mathbb{C}$ be regular and $X_t = h^{-1}(t)$. There is a triangle

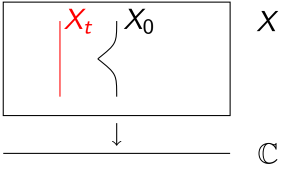
$$\imath^*\mathcal{E}|_{\mathrm{Re}(h)<0}[-1] \to \imath^* R\Gamma_{\mathrm{Re}(h)\geq 0}(\mathcal{E}) \to \imath^*\mathcal{E} \to \imath^*\mathcal{E}|_{\mathrm{Re}(h)<0}$$



Nearby and vanishing cycles ...

Let $h : X \to \mathbb{C}$ be regular and $X_t = h^{-1}(t)$. There is a triangle

$$^p\psi_h(\mathcal{E}) \to \imath^* R\Gamma_{\mathrm{Re}(h)\geq 0}(\mathcal{E}) \to \imath^*\mathcal{E} \to {}^p\psi_h(\mathcal{E})[1].$$



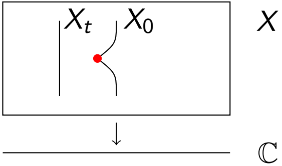
The nearby cycles $^p\psi_h(\mathcal{E})$ are related to the (local) Milnor fibre:

$$H^i\left({}^p\psi_h(\mathcal{E})\right)_x \cong H^i(MF_x; \mathcal{E}).$$

Nearby and vanishing cycles . . .

Let $h : X \to \mathbb{C}$ be regular and $X_t = h^{-1}(t)$. There is a triangle

$$ {}^p\psi_h(\mathcal{E}) \to {}^p\varphi_h(\mathcal{E}) \to \imath^*\mathcal{E} \to {}^p\psi_h(\mathcal{E})[1] $$



The nearby cycles ${}^p\psi_h(\mathcal{E})$ are related to the (local) Milnor fibre:

$$ H^i\left({}^p\psi_h(\mathcal{E})\right)_x \cong H^i(MF_x; \mathcal{E}). $$

The vanishing cycles ${}^p\varphi_h(\mathcal{E})$ are supported on $\mathrm{Crit}(h) \cap X_0$.
Normal Morse data is a special case: we can choose h (locally) so that

$$ \mathrm{NMD}(\mathcal{E}, S) \cong {}^p\varphi_h(\mathcal{E})_x[\dim S]. $$

Monodromy

'Rotating $\mathbb{C}$' induces monodromy maps μ on ${}^p\psi_h(\mathcal{E})$ and ${}^p\varphi_h(\mathcal{E})$.
Also have maps

$$ {}^p\psi_h(\mathcal{E}) \underset{v}{\overset{c}{\rightleftarrows}} {}^p\varphi_h(\mathcal{E}) $$

such that these monodromies are $1 + cv$ and $1 + vc$.
These induce maps between the unipotent parts

$$ {}^p\psi_h^{\mathrm{un}}(\mathcal{E}) \underset{v}{\overset{c}{\rightleftarrows}} {}^p\varphi_h^{\mathrm{un}}(\mathcal{E}) $$

(and isomorphisms between the non-unipotent parts).

and how to glue perverse sheaves

Theorem ([GM83b, KS90, Mas09])

${}^p\psi_h$ and ${}^p\varphi_h$ preserve perverse sheaves, and commute with duality.

Theorem ([Beĭ87])

The categories Perv(X) and Glue(X, h) are equivalent via

$$\mathcal{E} \mapsto \left(\mathcal{E}|_{X-X_0}, {}^p\varphi_h^{un}(\mathcal{E}), c, v\right).$$

Here Glue(X, h) is the category with objects $(\mathcal{E}, \mathcal{F}, c, v)$ where $\mathcal{E} \in$ Perv$(X - X_0)$ and $\mathcal{F} \in$ Perv(X_0) with

$$\mathcal{F} \xrightarrow{v} {}^p\psi_h^{un}(\mathcal{E}) \xrightarrow{c} \mathcal{F} \qquad \mu = 1 + vc$$

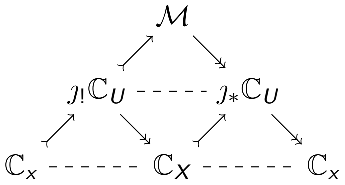
and morphisms given by commuting diagrams.

Quiver descriptions of perverse sheaves. . .

- $\mathrm{Perv}_{\mathbb{S}}(\mathbb{CP}^n)$ is equivalent to representations of the quiver

$$Q: \quad 0 \underset{q}{\overset{p}{\rightleftarrows}} 1 \underset{q}{\overset{p}{\rightleftarrows}} \cdots \underset{q}{\overset{p}{\rightleftarrows}} n$$

  with $1 + qp$ invertible and all other length two paths zero. E.g. when $n = 1$ the indecomposable perverse sheaves are



- $\mathrm{Perv}_{\mathbb{S}}(M_n(\mathbb{C}))$ is equivalent to representations of Q but with relations $pq = qp$ and $1 + pq$, $1 + qp$ invertible [BG99].

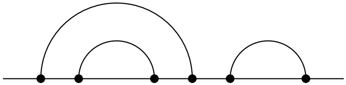
Theorem ([GMV96])

$\mathrm{Perv}_{\mathbb{S}}(X)$ admits a quiver description.

representation theory. . .

- Quiver description of $\mathrm{Perv}_{\mathbb{S}}(G/B)$ [Vyb07].
- Quiver description of $\mathrm{Perv}_{\mathbb{S}}(Gr_m(\mathbb{C}^{2m}))$ as A-modules [Bra02]. There is a diagrammatic description of A using fact that

$$\begin{array}{c}\text{Indecomp proj-inj}\\ \text{perverse sheaves}\end{array} \longleftrightarrow \begin{array}{c}\text{Crossingless matchings}\\ \text{of } 2m \text{ points}\end{array}$$



From these matchings Khovanov [Kho00] constructed

$$\mathcal{H}_m \cong \mathrm{End}\left(\bigoplus \text{indecomp proj-inj}\right)$$

Stroppel [Str09] generalised to a diagrammatic description

$$\mathcal{K}_m \cong \mathrm{End}\left(\bigoplus \text{indecomp proj}\right) \cong A.$$

and an intriguing invariant

Stroppel's description opens the possibility of computing

$$\mathrm{Ext}^*(\mathrm{IC}_{\overline{S}}(\mathbb{C}), \mathrm{IC}_{\overline{S}}(\mathbb{C}))$$

for a Schubert variety $\overline{S} \subset Gr_m(\mathbb{C}^{2m})$. This would yield interesting examples of the groups

$$\mathrm{Ext}^*(\mathrm{IC}_X(\mathbb{C}), \mathrm{IC}_X(\mathbb{C})).$$

These are

- topological invariants of X,
- isomorphic to $H^*(X)$ when X non-singular,
- graded rings over which $IH^*(X)$ is a graded module,
- subrings of $H^*(\widetilde{X})$ for any resolution $\widetilde{X}$.
- hard to compute!

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