Topological aspects of perverse sheaves

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Part I

Stratified spaces

Stratifications

A stratification of a topological space X consists of

1. a decomposition $X = \bigsqcup_i S_i$ into disjoint locally-closed subspaces
2. geometric conditions on the strata S_i
3. conditions on how the strata fit together

There are many variants of these conditions (topological, PL, smooth, analytic, algebraic) depending on context. We will work with a smooth version: Whitney stratifications.

Whitney stratifications

A locally-finite decomposition $M = \bigsqcup_i S_i$ of a smooth manifold $M \subset \mathbb{R}^N$ is a Whitney stratification if

1. each S_i is a smooth submanifold
2. the frontier condition holds: $S_i \cap \overline{S_j} \neq \emptyset \implies S_i \subset \overline{S_j}$
3. the Whitney B condition holds: for sequences (x_k) in S_i and (y_k) in S_j with $x_k, y_k \to x$ as $k \to \infty$ one has

$$\lim_{k\to\infty} \overline{x_k y_k} \subset \lim_{k\to\infty} T_{y_k} S_j$$

Remarks

- Whitney B independent of embedding $M \subset \mathbb{R}^N$
- Whitney B $\implies$ Whitney A: $T_x S_i \subset \lim_{k\to\infty} T_{y_k} S_j$

Whitney stratified spaces

A Whitney stratified space is a union of strata $X \subset M$ in a Whitney stratification of M.

Examples

- Manifold with marked submanifold
- Manifold with boundary $(M, \partial M)$
- $\mathbb{RP}^m$ or $\mathbb{CP}^m$ filtered by projective subspaces
- Whitney umbrella: $\{x^2 = y^2 z\} \subset \mathbb{R}^3$

Theorem (Whitney 1965)

A real or complex analytic variety admits a Whitney stratification by analytic subvarieties.

In fact, definable subsets of any o-minimal expansion of $\mathbb{R}$ admit Whitney stratifications, e.g. semi-algebraic or subanalytic subsets.

Local structure and stratified maps

A Whitney stratified space X admits the structure of a Thom–Mather stratification. In particular,

- the stratification is locally topologically trivial
- each stratum $S \subset X$ has a (topologically) well-defined link L such that each $x \in S$ has a neighbourhood stratum-preserving homeomorphic to

$$\mathbb{R}^{\dim S} \times C(L)$$

  where $C(L) = L \times [0,1)/L \times \{0\}$ is the cone on L.

A smooth map $f\colon X \to Y$ of Whitney stratified spaces is stratified if the preimage of each stratum of Y is a union of strata of X.

Theorem (Whitney 1965)

For proper, analytic $f\colon X \to Y$ one can refine stratifications of X and Y so that f is stratified.

Exit paths

Let $||\Delta^n||$ be the geometric n-simplex with 'strata'

$$S_i = \{(t_0, \ldots, t_n) \mid t_i \neq 0, t_{i+1} = \cdots = t_n = 0\} \qquad (0 \leq i \leq n)$$

For Whitney stratified X, consider continuous stratified maps

$$||\Delta^n|| \to X$$

The restriction to the 'spine' is an exit path; the restriction to the edge $[0n]$ is an elementary exit path.

Theorem (Nand-Lal–W. 2016, c.f. Millar 2013)

Let SSX be the simplicial set with $SSX_n = \{||\Delta^n|| \to X\}$. Then SSX is a quasi-category (spines can be completed to simplices).

Fundamental, or exit, category

The objects of $\tau_1 X$ are the points of X and the morphisms

$$\tau_1 X(x, y) = \{\text{elementary exit paths from } x \text{ to } y\}/\text{homotopy}$$

Composition is given by concatenation followed by deformation to an elementary exit path. For example, if X has one stratum then $\tau_1 X = \Pi_1 X$ is the fundamental groupoid.

Examples

- $\tau_1 ||\Delta^n|| \simeq \langle\ 0 \to 1 \to \cdots \to n\ \rangle$
- $\tau_1 (\{0\} \subset \mathbb{C}) \simeq \langle\ 0 \to 1 \circlearrowleft \mathbb{Z}\ \rangle$

The fundamental category is a functor — stratified $f\colon X \to Y$ induces $\tau_1 f\colon \tau_1 X \to \tau_1 Y$.

Local systems and covers

Let X be a topological space. Consider sheaves of k-vector spaces.

Definition (Local system)

Locally-constant sheaf on X with finite-dimensional stalks.

Theorem

For X locally 1-connected there are equivalences of categories

- $\mathrm{Cov}(X) \simeq \mathrm{Fun}(\Pi_1 X, Set)$
- $Loc(X; k) \simeq \mathrm{Fun}(\Pi_1 X, k\text{-}VS)$

Sketch proof.

Covers have unique path lifting for all paths. Similarly, local systems induce monodromy functors $\Pi_1 X \to k$-vs. □

Constructible sheaves and stratified étale covers

Let X be a Whitney stratified space.

Definition (Constructible sheaf)

Sheaf on X whose restriction to each stratum is a local system.

Definition (Stratified étale cover)

Étale map $p\colon Y \to X$ which restricts to a cover of each stratum.

Theorem (MacPherson 1990s, c.f. W. 2008)

For X Whitney stratified there are equivalences of categories

- $\mathrm{EtCov}(X) \simeq \mathrm{Fun}\,(\tau_1 X, \mathit{Set})$
- $\mathrm{Constr}(X;k) \simeq \mathrm{Fun}\,(\tau_1 X, k\text{-}\mathit{vs})$

Sketch proof.

Étale covers have unique path lifting for exit paths. Similarly, constructible sheaves induce monodromy functors $\tau_1 X \to k$-vs. □

Remarks and examples

Remarks

- There is a dual version — 'entry category' $\tau_1 X^{\mathrm{op}}$ classifies 'stratified branched covers' and 'constructible cosheaves'
- Functoriality of $\tau_1 X$ for stratified maps $f\colon X \to Y$ induces

$$\begin{aligned} \mathrm{EtCov}(Y) \to \mathrm{EtCov}(X) &: \quad Z \mapsto Y \times_X Z \\ \mathrm{Constr}(Y) \to \mathrm{Constr}(X) &: \quad \mathcal{E} \mapsto f^*\mathcal{E} \end{aligned}$$

Examples

- $\mathrm{Constr}(\{0\} \subset \mathbb{C})$ are representations of $0 \to 1 \circlearrowleft$
- $\mathrm{Constr}\big(\{0\} \subset \mathbb{CP}^1\big)$ are representations of $0 \to 1$

Part II

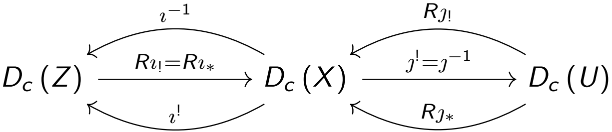
Perverse sheaves

Constructible derived category

- $\mathcal{E}^\bullet \in D_c(X) \iff \mathcal{H}^d(\mathcal{E}^\bullet) \in \mathrm{Constr}(X)$ for all $d \in \mathbb{Z}$
- Poincaré–Verdier duality $D_X \colon D_c(X)^{\mathrm{op}} \xrightarrow{\sim} D_c(X)$
- $\mathcal{E}^\bullet \in D_c(X)$ has finite-dimensional cohomology:

$$\mathbb{H}^d(X; \mathcal{E}^\bullet) = H^d(Rp_*\mathcal{E}^\bullet) \cong \mathrm{Hom}(k_X, \mathcal{E}^\bullet[d])$$

- for open $\jmath \colon U \hookrightarrow X$ and closed $\imath \colon Z = X - U \hookrightarrow X$ have



giving rise to (dual) natural exact triangles:

$$R\imath_!\imath^!\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to R\jmath_*\jmath^{-1}\mathcal{E}^\bullet \to R\imath_!\imath^!\mathcal{E}^\bullet[1]$$
$$R\jmath_!\jmath^!\mathcal{E}^\bullet \to \mathcal{E}^\bullet \to R\imath_*\imath^{-1}\mathcal{E}^\bullet \to R\jmath_!\jmath^!\mathcal{E}^\bullet[1]$$

Cohomology of local systems

Let M be an oriented (real) manifold and $\mathcal{L} \in \mathrm{Loc}(M)$. Then

- $\mathbb{H}^d(M; \mathcal{L}) = 0$ for $d < 0$ and $d > \dim M$
- $\chi(M; \mathcal{L}) = \dim(\mathcal{L})\chi(M)$

Remarks

- The vanishing result follows from the isomorphism

$$D_M \mathcal{L} \cong \mathcal{L}^\vee[\dim M]$$

  which implies $\mathbb{H}_c^d(M; \mathcal{L}) \cong \mathbb{H}^{\dim M - d}(M; \mathcal{L}^\vee)^\vee$

- The second fact generalises the formula

$$\chi(E) = \chi(B)\chi(F)$$

  for a fibration $F \to E \to B$.

Example: local systems on $\mathbb{C}^*$

Consider an n-dimensional $\mathcal{L} \in \mathsf{Loc}(\mathbb{C}^*)$ as a representation

$$\pi_1 \mathbb{C}^* \to \mathrm{GL}_n(k)$$

and let $\mu_{\mathcal{L}}$ denote the image of the generator. Then

$$\mathbb{H}^d(\mathbb{C}^*; \mathcal{L}) = \begin{cases} \ker(\mu_{\mathcal{L}} - 1) & d = 0 \\ \operatorname{coker}(\mu_{\mathcal{L}} - 1) & d = 1 \\ 0 & d \neq 0, 1 \end{cases}$$

Identifying $\mathbb{C}^*$ with $\{xy = 1\} \subset \mathbb{C}^2$ exhibits the vanishing for $d > 1$ as an example of

Theorem (Artin vanishing for local systems)

If M is a smooth affine complex variety then

$$\mathbb{H}^d(X; \mathcal{L}) = 0 \quad \textit{for } d > \dim_{\mathbb{C}} M$$
$$\mathbb{H}^d_c(X; \mathcal{L}) = 0 \quad \textit{for } d < \dim_{\mathbb{C}} M$$

From local systems to perverse sheaves

Constructible sheaves are a special case of perverse sheaves:

- $\mathrm{Constr}(X)$ is 'glued' from local systems on the strata
- Perverse sheaves are 'glued' from shifted local systems

Lemma

$\mathrm{Constr}(X) \hookrightarrow D_c(X)$ is a full abelian subcategory with $D_c(X)$ as its triangulated closure.

Example ($X = \mathbb{CP}^1$)

$\mathrm{Constr}(X) \simeq k$-vs so $\mathrm{Hom}_{D_b\mathrm{Constr}(X)}(k_X, k_X[d]) = 0$ for $d \neq 0$ but

$$\mathrm{Hom}_{D_c(X)}(k_X, k_X[2]) \cong H^2(X; k) \cong k$$

This shows $D_c(X) \not\simeq D^b\mathrm{Constr}(X)$ in general.

Truncation structures

A *t*-structure $D_c^{\leq 0}(X) \subset D_c(X)$ is an ext-closed subcategory with

- $D_c^{\leq 0}(X)[1] \subset D_c^{\leq 0}(X)$
- every $\mathcal{E}^\bullet \in D_c(X)$ sits in a triangle

$$\mathcal{D}^\bullet \to \mathcal{E}^\bullet \to \mathcal{F}^\bullet \to \mathcal{D}^\bullet[1]$$

with $\mathcal{D}^\bullet \in D_c^{\leq 0}(X)$ and $\mathcal{F}^\bullet \in D_c^{\geq 1}(X) = D_c^{\leq 0}(X)^\perp$

The *t*-structure is bounded if

$$D_c(X) = \bigcup_{n \in \mathbb{N}} D_c^{\geq -n}(X) \cap D_c^{\leq n}(X)$$

where $D_c^{\leq n}(X) = D_c^{\leq 0}(X)[-n]$ etc.

Example (Standard *t*-structure)

$$D_c^{\leq 0}(X) = \{\mathcal{E}^\bullet \mid \mathcal{H}^i\mathcal{E} = 0 \text{ for } i > 0\}$$

Hearts and cohomology

Theorem (Beilinson, Bernstein, Deligne 1982)

- $D_c^{\leq 0}(X) \hookrightarrow D_c(X)$ *has a right adjoint* $\tau^{\leq 0}$
- $D_c^{\geq 0}(X) \hookrightarrow D_c(X)$ *has a left adjoint* $\tau^{\geq 0}$
- *heart* $D_c^0(X) = D_c^{\leq 0}(X) \cap D_c^{\geq 0}(X)$ *is an abelian subcategory*
- $\mathcal{H}^0 = \tau^{\leq 0}\tau^{\geq 0} \colon D_c(X) \to D_c^0(X)$ *is cohomological*

Example

The heart of the standard t-structure is $\mathrm{Constr}(X)$, and $\mathcal{H}^0$ and $\tau^{\leq 0}$ are the previously defined functors.

Remark (heart determines a bounded t-structure)

$$D_c^{\leq 0}(X) = \langle D_c^0(X), D_c^0(X)[1], \ldots \rangle$$

Glueing t-structures

The most important way of constructing t-structures (for us) is via the following glueing construction. Suppose $\jmath\colon U \hookrightarrow X$ is an open union of strata and $\imath\colon Z \hookrightarrow X$ the complementary closed inclusion.

Theorem (Beilinson, Bernstein, Deligne 1982)

Given t-structures $D_c^{\leq 0}(U)$ and $D_c^{\leq 0}(Z)$ there is a unique 'glued' t-structure $D_c^{\leq 0}(X)$ such that

$$\mathcal{E}^\bullet \in D_c^{\leq 0}(X) \iff \jmath^{-1}\mathcal{E}^\bullet \in D_c^{\leq 0}(U) \text{ and } \imath^{-1}\mathcal{E}^\bullet \in D_c^{\leq 0}(Z)$$

dually $\mathcal{E}^\bullet \in D_c^{\geq 0}(X) \iff \jmath^{-1}\mathcal{E}^\bullet \in D_c^{\geq 0}(U)$ *and* $\imath^{!}\mathcal{E}^\bullet \in D_c^{\geq 0}(Z)$.

Example (Standard t-structure)

The t-structure with heart $\mathrm{Constr}(X)$ is glued from those with hearts $\mathrm{Constr}(U)$ and $\mathrm{Constr}(Z)$, hence inductively from those on $D_c(S)$ with heart $\mathrm{Loc}(S)$ for each stratum $S \subset X$.

Perverse sheaves

Let X be Whitney stratified. Fix a perversity, i.e. $p\colon \mathbb{N} \to \mathbb{Z}$ with $p(0) = 0$ and

$$m \leq n \implies 0 \leq p(m) - p(n) \leq n - m$$

Inductively glueing the t-structures in $D_c(S)$ with hearts

$$\mathrm{Loc}(S)[-p(\dim S)] \quad \text{for strata } S \subset X$$

gives t-structure with heart the p-perverse sheaves ${}^p\mathrm{Perv}(X)$. Let $\imath_S\colon S \hookrightarrow X$. Perverse sheaves are characterised by

$$\mathcal{E}^\bullet \in {}^p\mathrm{Perv}(X) \iff \begin{cases} \mathcal{H}^i\left(\imath_S^{-1}\mathcal{E}^\bullet\right) = 0 & \text{for } i > p(\dim S) \\ \mathcal{H}^i\left(\imath_S^{!}\mathcal{E}^\bullet\right) = 0 & \text{for } i < p(\dim S) \end{cases}$$

Example

X smooth with one stratum $\implies$ ${}^p\mathrm{Perv}(X) = \mathrm{Loc}(X)[-p(\dim X)]$

Intermediate extensions

Let $\imath\colon Z \hookrightarrow X$ be the inclusion of a closed union of strata. Then

$$\mathcal{E}^\bullet \in {}^p\mathrm{Perv}(Z) \implies R\imath_*\mathcal{E}^\bullet \cong R\imath_!\mathcal{E}^\bullet \in {}^p\mathrm{Perv}(X)$$

For the complementary open inclusion $\jmath\colon X - Z \hookrightarrow X$ we only have

$$\mathcal{E}^\bullet \in {}^p\mathrm{Perv}(X-Z) \implies R\jmath_*\mathcal{E}^\bullet \in {}^pD_c^{\geq 0}(X) \text{ and } R\jmath_!\mathcal{E}^\bullet \in {}^pD_c^{\leq 0}(X)$$

The intermediate extension is the perverse sheaf defined by

$$^p\jmath_{!*}\mathcal{E}^\bullet = \operatorname{im} {}^p\mathcal{H}^0(R\jmath_!\mathcal{E}^\bullet \to R\jmath_*\mathcal{E}^\bullet)$$

Proposition

For strata $S \subset Z$ the intermediate extension $^p\jmath_{!*}\mathcal{E}^\bullet$ satisfies

$$\begin{cases} \mathcal{H}^i\left(\imath_S^{-1}\,{}^p\jmath_{!*}\mathcal{E}^\bullet\right) = 0 & \text{for } i \geq p(\dim S) \\ \mathcal{H}^i\left(\imath_S^{!}\,{}^p\jmath_{!*}\mathcal{E}^\bullet\right) = 0 & \text{for } i \leq p(\dim S) \end{cases}$$

and has no subobjects or quotients supported on Z.

Properties of perverse sheaves

Proposition

Let $\mathcal{E}^\bullet \in {}^pPerv(X)$ *be a perverse sheaf. Then*

- $\mathbb{H}^d(X; \mathcal{E}^\bullet) \neq 0 \implies p(\dim X) \leq d \leq \dim X + p(\dim X)$
- $\mathcal{E}^\bullet$ *is simple* $\iff \mathcal{E}^\bullet \cong {}^p\jmath_{!*}\mathcal{L}[-\dim S]$ *for irreducible* $\mathcal{L} \in Loc(S)$ *where* $\jmath \colon S \hookrightarrow \overline{S}$ *is the inclusion*

Proposition

The category ${}^pPerv(X)$ *has many nice properties:*

- *it is a stack*
- *it is Artinian and Noetherian*
- *it is Krull–Remak–Schmidt*
- *duality induces an equivalence*

$$D_X \colon {}^pPerv(X)^{op} \xrightarrow{\sim} {}^{p^*}Perv(X)$$

Example: perverse sheaves on $X = (\{0\} \subset \mathbb{C})$

Fix $k = \mathbb{C}$. Let $\mathcal{L}_\mu \in \mathrm{Loc}(\mathbb{C}^*)$ have rank 1 and monodromy $\mu \in \mathbb{C}^*$. The simple perverse sheaves in ${}^m\mathrm{Perv}(X)$ are

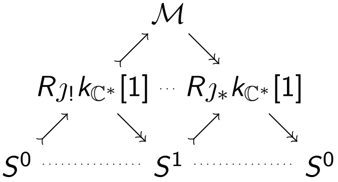
$$S^\mu = \jmath_{!*}\mathcal{L}_\mu[1] \text{ and } S^0 = R\imath_* k_0$$

The only non-zero Ext-groups are, for $\mu \neq 0$,

$$\mathrm{Ext}^1(S^\mu, S^\mu) \cong \mathrm{Ext}^1(S^1, S^0) \cong \mathrm{Ext}^1(S^0, S^1) \cong k$$

Hence ${}^m\mathrm{Perv}(X) = \langle S^0, S^1\rangle \oplus \bigoplus_{\mu \neq 0,1} \langle S^\mu \rangle$ with indecomps in

- $\langle S^\mu \rangle$ corresponding to Jordan blocks J_n^μ
- $\langle S^0, S^1 \rangle$ corresponding to one of four extensions of J_n^1, e.g.



Example: maps between smooth curves

Suppose $f\colon X \to Y$ is a map between smooth curves. Then

$$Rf_* k_X[1] \cong \jmath_{!*}\mathcal{L}[1]$$

where $\jmath\colon U \to Y$ is the smooth locus and $\mathcal{L} = \jmath^{-1} Rf_* k_X$.

Remark (Instance of Decomposition Theorem)

When $k = \mathbb{C}$ the perverse sheaf $\jmath_{!*}\mathcal{L}[1]$ is semi-simple.

Example

Let $f\colon X \to \mathbb{CP}^1$ be a smooth hyper-elliptic curve of genus g ramified at $2(g+1)$ points. The monodromy of $\mathcal{L}$ at each is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $Rf_* k_X[1] \cong \mathbb{C}_{\mathbb{CP}^1} \oplus R\jmath_! \mathcal{M}[1]$ where $\mathcal{M}$ is rank 1 local system with monodromy -1 at each ramification point.

Example: stratifications with finite fundamental groups

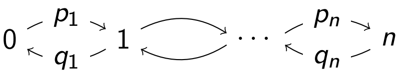
Theorem (Cipriani–W. 2017)

Suppose $\pi_1 S$ finite for all strata $S \subset X$. Then ${}^p Perv(X)$

- *has finitely many simple objects*
- *has enough projectives and enough injectives*
- ${}^p Perv(X) \simeq \mathrm{Rep}(\mathrm{End}\,\mathcal{P}^\bullet)$ *for projective generator $\mathcal{P}^\bullet$*

Example

Middle perversity perverse sheaves on $\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \cdots \subset \mathbb{CP}^n$ are representations of



with $1 - q_1 p_1$ invertible and all other length two paths zero.

Intersection cohomology

The intersection cohomology complex associated to $\mathcal{L} \in \mathrm{Loc}(S)$ is

$$^{p}\mathcal{IC}^{\bullet}(\mathcal{L}) = {^{p}\jmath_{!*}}\mathcal{L}[-p(\dim S)] \in {^{p}\mathrm{Perv}}(X)$$

where $\jmath \colon S \to \overline{S}$. The associated intersection cohomology is

$$^{p}IH^{*}(X;\mathcal{L}) = \mathbb{H}^{*+p(\dim S)}\left(X; {^{p}\mathcal{IC}^{\bullet}}(\mathcal{L})\right).$$

Theorem (Poincaré duality)

There is an isomorphism

$$D_X {^{p}\mathcal{IC}^{\bullet}}(\mathcal{L}) \cong {^{p^*}\mathcal{IC}^{\bullet}}(\mathcal{L}^{\vee})$$

It follows that $^{p}IH_c^{d}(X;\mathcal{L}) \cong {^{p^*}IH^{\dim S - d}}(X;\mathcal{L}^{\vee})$

Comparing perversities with classical perversities

Suppose $x \in S' \subset \overline{S} - S$ and L is the link of S' in $\overline{S}$ and $\mathcal{L} \in \mathrm{Loc}(S)$. Then the stalk cohomology $\mathcal{H}^d_x({}^p\mathcal{IC}^\bullet(\mathcal{L}))$ is

$$ {}^pIH^{d-p(\dim S)}(C(L);\mathcal{L}) \cong \begin{cases} {}^pIH^{d-p(\dim S)}(L;\mathcal{L}) & d < p(\dim S') \\ 0 & d \geq p(\dim S') \end{cases} $$

Since $\dim S' < \dim S$ and p is a decreasing function:

$$ {}^p\mathcal{IC}^\bullet(\mathcal{L}) \cong \cdots \tau_{<p(\dim S')} Rj_{S'*} \cdots \tau_{\leq p(\dim S)} \mathcal{L}[-p(\dim S)]. $$

Comparing with Deligne's formula for the classical perversity $\overline{p}$

$$ {}^{\overline{p}}\mathcal{IC}^\bullet(k_U) \cong \cdots \tau_{\leq \overline{p}(\mathrm{codim}\, S') - n} Rj_{S'*} \cdots \tau_{\leq -n} k_U[n] $$

(where $U \subset X$ open and $\dim X = 2n$) we deduce that

$$ \overline{p}(2n-d) = \begin{cases} p(d) - p(2n) - 1 & d < 2n \\ 0 & d = 2n \end{cases} $$

Families of stratifications

Let $\mathbb{S}$ be a family of Whitney stratifications of X, such that any two admit a common refinement. For example $\mathbb{S}$ might consist of all semialgebraic stratifications, or all stratifications by analytic or algebraic varieties. The $\mathbb{S}$-constructible derived category is

$$D_{\mathbb{S}-c}(X) = \operatorname{colim}_{\mathcal{S} \in \mathbb{S}} D_c(X_{\mathcal{S}})$$

and similarly ${}^p\mathrm{Perv}_{\mathbb{S}-c}(X) = \operatorname{colim}_{\mathcal{S} \in \mathbb{S}} {}^p\mathrm{Perv}(X_{\mathcal{S}})$.

Theorem (Beilinson 1987)

$D_{alg-c}(X) \simeq D^{b\,m}Perv_{alg-c}(X)$ *where* $m(d) = -d/2$

Theorem (Kashiwara–Schapira 1990)

$D_{\mathbb{R}-an-c}(X) \simeq D^b\mathrm{Constr}_{\mathbb{R}-an-c}(X)$

Part III

Morse theory

Classical Morse theory

Let M be a compact, oriented manifold. Say $f : M \to \mathbb{R}$ is Morse if it has only non-degenerate critical points, equivalently if

$$\Gamma_{df} \pitchfork T^*_M M \subset T^* M$$

where $T^*_S M = \{(x, \alpha) \in T^* M \mid \alpha|_{T_x S} = 0\}$ for smooth $S \subset M$.

Lemma (Cohomological Morse Lemma)

If there is one critical point $x \in f^{-1}[a, b)$ then there is a LES

$$\cdots \to k[-\mathrm{ind}_x f] \to H^*(X_{<b}; k) \to H^*(X_{<a}; k) \to \cdots$$

Corollary (Index or Poincaré–Hopf Theorem)

Relating indices to orientations of intersections we obtain

$$\chi(M) = \Gamma_{df} \cdot T^*_M M = T^*_M M \cdot T^*_M M$$

Stratified Morse functions

Let $X \subset M$ be Whitney stratified. Then the conormal space

$$T^*_X M = \bigcup_{S \subset X} T^*_S M$$

is closed in T^*M. A covector in T^*M is degenerate if it lies in

$$\bigcup_{S \subset X} \left(\overline{T^*_S M} - T^*_S M \right)$$

i.e. if it vanishes on a generalised tangent space.

Definition (Stratified Morse function)

Smooth $f \colon X \to \mathbb{R}$ whose restriction $f|_S$ to each stratum $S \subset X$ is Morse with df non-degenerate at each critical point; equivalently if

$$\Gamma_{df} \pitchfork T^*_S M \quad \text{and} \quad \Gamma_{df} \cap \left(\overline{T^*_S M} - T^*_S M \right) = \emptyset$$

for each stratum $S \subset X$.

Morse data

Let $x \in S \subset X$ and N be a normal slice to S at x in M. Let

$$\imath \colon X_{\geq c} \hookrightarrow X \quad \text{and} \quad \imath_N \colon N \cap X_{\geq c} \hookrightarrow N \cap X$$

The local Morse data and normal Morse data of $\mathcal{E}^\bullet \in D_c(X)$ are

$$\mathrm{LMD}(\mathcal{E}^\bullet, f, x) = \left(\imath^! \mathcal{E}^\bullet\right)_x \quad \text{and} \quad \mathrm{NMD}(\mathcal{E}^\bullet, f, x) = \left(\imath_N^! \mathcal{E}^\bullet\right)_x$$

Proposition

If $d_x(f|_S) \neq 0$ *then* $\mathrm{LMD}(\mathcal{E}^\bullet, f, x) \cong 0 \cong \mathrm{NMD}(\mathcal{E}^\bullet, f, x)$. *If* $d_x(f|_S) = 0$ *then*

$$\mathrm{LMD}(\mathcal{E}^\bullet, f, x) \cong \mathrm{NMD}(\mathcal{E}^\bullet, f, x)[-\mathrm{ind}_x f|_S]$$

and $\mathrm{NMD}(\mathcal{E}^\bullet, f, x)$ *depends only on the component of* $d_x f$ *in the non-degenerate covectors* $T_S^* M - \bigcup_{S' > S} \overline{T_{S'}^* M}$

Examples of Morse data

Example (Local system $\mathcal{L} \in \mathrm{Loc}(X)$ and $x \in S$)

- $\operatorname{codim} S = 0 \implies \mathrm{NMD}(\mathcal{L}, f, x) = \mathcal{L}_x$
- $\operatorname{codim} S > 0$ and X smooth $\implies \mathrm{NMD}(\mathcal{L}, f, x) = 0$

Example (X a complex curve, Σ singular set)

For any stratified Morse function $f \colon X \to \mathbb{R}$

$$\mathrm{NMD}(k_X, f, x) = \begin{cases} k^{m_x - 1}[-1] & x \in \Sigma \\ k & x \notin \Sigma \end{cases}$$

$$\mathrm{NMD}({}^m\mathcal{IC}^\bullet(k_{X-\Sigma}), f, x) = \begin{cases} k^{m_x - b_x} & x \in \Sigma \\ k[1] & x \notin \Sigma \end{cases}$$

where m_x is the multiplicity and b_x the number of branches.

Morse theory for constructible complexes

Lemma (Cohomological Morse Lemma II)

If there is one critical point $x \in f^{-1}[a, b)$ then there is a LES

$$\cdots \to \mathrm{NMD}(\mathcal{E}^\bullet, f, x)[-\mathrm{ind}_x f|_S] \to \mathbb{H}^*(X_{<b}; \mathcal{E}^\bullet) \to \mathbb{H}^*(X_{<a}; \mathcal{E}^\bullet) \to \cdots$$

Example (Pinched torus / nodal cubic)

Let $X = \{(x, y, z) \in \mathbb{CP}^2 \mid x^3 + y^3 = xyz\}$ be the nodal cubic. Then

$$H^i(X; k) \cong \begin{cases} k & i = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$^m IH^i(X; k) \cong \begin{cases} k & i = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

Complex stratified Morse theory

Suppose $X \subset M$ is a complex analytic Whitney stratified space. Then for critical $x \in S$

$$\mathrm{NMD}(\mathcal{E}^\bullet, f, x) = \mathrm{NMD}(\mathcal{E}^\bullet, S)$$

depends only on S.

Corollary (Brylinski–Dubson–Kashiwara Index Theorem 1981)

Carefully choosing orientations to compute the intersection we obtain

$$\chi(X; \mathcal{E}^\bullet) = \Gamma_{df} \cdot CC(\mathcal{E}^\bullet) = T^*_M M \cdot CC(\mathcal{E}^\bullet)$$

where

$$CC(\mathcal{E}^\bullet) = \sum_S (-1)^{\dim_{\mathbb{C}} S} \chi\left(\mathrm{NMD}(\mathcal{E}^\bullet, S)\right) T^*_S M$$

is the characteristic cycle of $\mathcal{E}^\bullet$*.*

Properties of characteristic cycles

- $\mathrm{CC}(\mathcal{E}^\bullet[1]) = -\mathrm{CC}(\mathcal{E}^\bullet)$
- For a triangle $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{E}^\bullet[1]$ one has

$$\mathrm{CC}(\mathcal{F}^\bullet) = \mathrm{CC}(\mathcal{E}^\bullet) + \mathrm{CC}(\mathcal{G}^\bullet)$$

- $\mathrm{CC}(D\mathcal{E}^\bullet) = \mathrm{CC}(\mathcal{E}^\bullet)$
- $\mathcal{E}^\bullet \in {}^m\mathrm{Perv}(X) \implies \mathrm{CC}(\mathcal{E}^\bullet)$ effective (see later)

Examples

- For a local system $\mathcal{L}$ on a closed stratum S

$$\mathrm{CC}(\mathcal{L}) = (-1)^{\dim_{\mathbb{C}} S}\mathrm{rank}(\mathcal{L})\, T_S^*M$$

  so that $\chi(S;\mathcal{L}) = \mathrm{rank}(\mathcal{L})\,\chi(S)$
- Characteristic cycles for ${}^m\mathrm{Perv}(\{0\} \subset \mathbb{CP}^1)$

Characteristic cycles for curves

If $X \subset M$ is a complex curve with singular set Σ then

$$\mathrm{CC}\,(k_X) = -\overline{T^*_{X-\Sigma}M} - \sum_{x\in\Sigma}(m_x - 1)\,T^*_x M$$
$$\mathrm{CC}\,(\mathcal{IC}^\bullet(k_{X-\Sigma})) = \overline{T^*_{X-\Sigma}M} + \sum_{x\in\Sigma}(m_x - b_x)\,T^*_x M$$

Hence

$$\mathrm{CC}\,(k_X) + \mathrm{CC}\,(\mathcal{IC}^\bullet(k_{X-\Sigma})) = \sum_{x\in\Sigma}(1 - b_x)\,T^*_x M$$

and so by the index theorem

$$\chi(X) - I\chi(X) = \sum_{x\in\Sigma}(1 - b_x)$$

Part IV

Special results for the middle perversity

Purity and perversity

Let X be a complex variety.

Definition (Purity)

$\mathcal{E}^\bullet$ is pure if $\mathrm{NMD}(\mathcal{E}^\bullet, S)$ is concentrated in degree $-\dim_{\mathbb{C}} S$.

Lemma

If $\mathcal{E}^\bullet$ is pure and $x \in S$ is only critical point in $f^{-1}[a,b)$ then

$$\mathbb{H}^d(X_{<b}, X_{<a}; \mathcal{E}^\bullet) \cong 0 \text{ for } d \neq \mathrm{ind}_x f|_S - \dim_{\mathbb{C}} S$$

In particular $\mathbb{H}^d(X; \mathcal{E}^\bullet) = 0$ *for* $|d| > \dim_{\mathbb{C}} X$

Theorem (Kashiwara–Schapira 1990)

Let $m(d) = -d/2$ be the middle perversity. Then

$$\mathcal{E}^\bullet \in {}^m Perv(X) \iff \mathcal{E} \text{ is pure}$$

Artin vanishing and consequences

Theorem (Perverse Artin vanishing)

If X is affine and $\mathcal{E}^\bullet \in {}^m Perv(X)$ then

$$\mathbb{H}^d(X;\mathcal{E}^\bullet) = 0 \text{ for } d > 0 \quad \text{and} \quad \mathbb{H}^d_c(X;\mathcal{E}^\bullet) = 0 \text{ for } d < 0$$

Corollary

If $f : X \to Y$ is affine and $\mathcal{E}^\bullet \in {}^m Perv(X)$ then

$$Rf_*\mathcal{E}^\bullet \in D_c^{\leq 0}(Y) \quad \text{and} \quad Rf_!\mathcal{E}^\bullet \in D_c^{\geq 0}(Y)$$

Corollary (Affine inclusions preserve perverse sheaves)

If $\jmath : X \hookrightarrow Y$ is an open affine inclusion then

$$R\jmath_*, R\jmath_! : {}^m Perv(X) \to {}^m Perv(Y)$$

Lefschetz Hyperplane Theorem

Theorem

Let $X \subset \mathbb{CP}^n$ be a complex projective variety, and H a generic hyperplane. Then the restriction

$$^{m}IH^{d}(X) \to {}^{m}IH^{d}(X \cap H)$$

is isomorphism for $d < \dim_{\mathbb{C}} X - 1$, injective for $d = \dim_{\mathbb{C}} X - 1$.

Example ($X = \{yz = 0\} \subset \mathbb{CP}^2$ and $H = \{x + y + z = 0\}$)

Since $|X \cap H| = 2$ the LHT $\implies \dim {}^{m}IH^{0}(X) \leq 2$. From the index theorem $I^{m}\chi(X) = 4$. Using Poincaré duality we see that

$$^{m}IH^{d}(X) \cong \begin{cases} k^2 & d = 0, 2 \\ 0 & d = 1 \end{cases}$$