

NUMERICALLY TRIVIAL FOLIATIONS, IITAKA FIBRATIONS AND THE NUMERICAL DIMENSION

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ABSTRACT. Modifying the notion of numerically trivial foliation of a pseudo-effective line bundle L introduced by the author in [Eck04a] (see also math.AG/0304312) it can be shown that the leaves of this foliation have codimension bigger than or equal to the numerical dimension of L , in the sense of Boucksom, Demailly, Paun and Peternell, math.AG/0405285. Furthermore, if the Kodaira dimension of L equal its numerical dimension the Kodaira-Iitaka fibration is its numerically trivial foliation. Both statements together yield a sufficient criterion for L not being abundant.

0. INTRODUCTION

In their seminal paper [BDPP04] Boucksom, Demailly, Paun and Peternell introduced a numerical dimension for pseudo-effective $(1, 1)$ -classes on compact Kähler manifolds generalizing the numerical dimension of nef line bundles on projective manifolds. For this purpose they used Boucksom's moving intersection numbers [Bou02] which can be defined as follows:

Definition 0.1. *Let X be a compact Kähler manifold with Kähler form ω . Let $\alpha_1, \dots, \alpha_p \in H^{1,1}(X, \mathbb{R})$ be pseudo-effective classes and let Θ be a closed positive current of bidimension (p, p) . Then the moving intersection number $(\alpha_1 \dots \alpha_p \cdot \Theta)_{\geq 0}$ of the α_i and Θ is defined to be the limit when $\epsilon > 0$ goes to 0 of*

$$\sup \int_{X-F} (T_1 + \epsilon\omega) \wedge \dots \wedge (T_p + \epsilon\omega) \wedge \Theta$$

where the T_i 's run through all currents with analytic singularities in $\alpha_i[-\epsilon\omega]$, and F is the union of the $\text{Sing}(T_i)$.

This may be used for

Definition 0.2. *Let X be an n -dimensional compact Kähler manifold. Then the numerical dimension $\nu(\alpha)$ of a pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$ is defined as*

$$\max\{k \in \{0, \dots, n\} : (\alpha^k \cdot \omega^{n-k})_{\geq 0} > 0\}$$

for some (and hence all) Kähler classes ω .

A pseudo-effective line bundle L is big iff $\nu(L) = \nu(c_1(L)) = n$ ([Bou02, Thm.3.1.31]). By cutting down with ample hypersurfaces this shows that the numerical dimension of the first Chern class of a pseudo-effective line bundle L is $\geq \kappa(X, L)$, the Kodaira-Iitaka dimension of L .

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Proving that on every projective complex manifold which is not uniruled the canonical bundle is pseudo-effective ([BDPP04, Cor.0.3]) Boucksom, Demailly, Paun and Peternell were able to use this notion of numerical dimension for generalizing the Abundance conjecture to

Conjecture 0.3. *On every projective manifold which is not uniruled we have*

$$\kappa(X) = \nu(X) = \nu(c_1(K_X)).$$

The author in turn tried to find more geometric obstacles for equality of Kodaira dimension and numerical dimension. In [Eck04a] this led him to the notion of numerically trivial foliations. The starting point is

Definition 0.4. *Let X be a compact Kähler manifold with Kähler form ω and pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$. A submanifold $Y \subset X$ (closed or not) is **numerically trivial** w.r.t. α iff for every immersed disk $\Delta \subset Y$,*

$$\limsup_{\epsilon \downarrow 0} \int_T \int_{\Delta' - \text{Sing } T} (T + \epsilon\omega) = 0,$$

where the T 's run through all currents with analytic singularities in $\alpha[-\epsilon\omega]$ and $\Delta' = \{t : |t| < 1 - \delta\}$ is any smaller disk contained in $\Delta = \{t : |t| < 1\}$.

This definition is applied to the leaves of a foliation on X which is allowed to have singularities. Such a foliation is given by a saturated subsheaf

$$\mathcal{F} \subset T_X$$

of the tangent bundle T_X which is closed under the Lie bracket. The singularities of \mathcal{F} form the analytic subset Z of points where

$$\mathcal{F}/m_{X,x}\mathcal{F} \rightarrow T_{X,x}$$

is not injective. By the Frobenius integrability theorem we can cover $X - Z$ by open sets $U_i \cong \Delta^n$ such that there exists smooth holomorphic maps $p_i : U_i \rightarrow \Delta^{n-k}$ induced by the projection $\Delta^n \rightarrow \Delta^{n-k}$ with

$$\mathcal{F}|_{U_i} = T_{U_i/\Delta^{n-k}}.$$

Further properties of singular foliations and constructions as the union of two foliations will be discussed in section 1.

Definition 0.5. *Let X be a compact Kähler manifold with a pseudo-effective class $\alpha \in H^{1,1}(X, \mathbb{R})$. A foliation $\{\mathcal{F}, (U_i, p_i)\}$ is **numerically trivial** w.r.t. α iff*

- (i) *every fiber of p_i is numerically trivial w.r.t. α ,*
- (ii) *and if $\Delta^2 \hookrightarrow U_i$ is an immersion such that the projection onto the first coordinate coincides with the projection $p_i : U_i \rightarrow \Delta^{n-k}$, then for any $\Delta' \subset \subset \Delta$ and any sequence of currents $T_k \in \alpha[-\epsilon_k\omega]$, $\epsilon_k \rightarrow 0$, the integrals $\int_{(\{z_1=a\} \cap \Delta') - \text{Sing } T_k} (T_k + \epsilon_k\omega)$ are uniformly (in a) bounded from above.*

By proving the Local Key Lemma [Eck04a, Lem.3.8] the author showed that it is possible to construct a numerically trivial foliation maximal w.r.t. inclusion which is called **the numerically trivial foliation of the pseudo-effective class α** . Furthermore, if α is the first Chern class $c_1(L)$ of a pseudo-effective line bundle L on X it is shown that the fibers of the Kodaira-Iitaka fibration ($m \gg 0$)

$$\phi|_{mL} : X \dashrightarrow Y$$

contain the leaves of the numerically trivial foliation of L ([Eck04a, 3.3]). Hence the dimension of its leaves is \leq than the dimension of the (generic) fibers of $\phi_{|mL|}$ and the codimension of the leaves is \geq than

$$\dim Y = \kappa(X).$$

When trying to compare the *numerical* dimension with the codimension of the leaves the author discovered that Def. 0.4 is not appropriate for this purpose. The point is that numerical dimensions are defined via integrals over n -dimensional complex manifolds whereas Def. 0.4 only uses integrals over 1-dimensional disks. Hence the usual difficulties when comparing L^p -integrable functions for different p occur. In the end this led the author to change the definition of numerical triviality:

Definition 0.6. *Let X be a compact Kähler manifold with Kähler form ω and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudo-effective $(1, 1)$ -class. A foliation \mathcal{F} is called numerically trivial iff for all $1 \leq p \leq n - 1$ and for all test forms $u \in \mathcal{D}^{(n-p, n-p)}(X - \text{Sing } \mathcal{F})$*

$$(NT)_u \quad \lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_X |(T_{ac} + \epsilon\omega)^p \wedge u| = 0$$

where the T 's run through all currents with analytic singularities representing α with $T \geq -\epsilon\omega$ and T_{ac} is the absolute continuous part of T in the Lebesgue decomposition.

Here a test form for \mathcal{F} is a smooth $(n-p, n-p)$ form with compact support outside the singularities of \mathcal{F} whose wedge product with every (p, p) form in $\Lambda^{p,p}(T_X^*/T_{\mathcal{F}}^*)$ is 0 (see section 2 for details).

At least in the surface case this definition of numerical triviality is implied by that in [Eck04a]. It is also possible to construct a maximal numerical trivial foliation w.r.t. the new definition, see section 2. Furthermore the proof of the following theorem becomes quite simple:

Theorem 0.7. *Let X be a compact Kähler manifold with Kähler form ω and $\alpha \in H^{1,1}(X, \mathbb{R})$ a pseudo-effective class. Let \mathcal{F} be the numerically trivial foliation of α . Then the numerical dimension $\nu(\alpha)$ is less or equal to the codimension of the leaves of \mathcal{F} .*

And a transversality criterion for detecting numerically trivial foliations (see Thm. 3.1) allows to show the second aim of this note:

Theorem 0.8. *Let X be a Kähler manifold and L a pseudo-effective line bundle on X . Suppose that the Kodaira-Iitaka dimension $\kappa(X, L)$ equals the numerical dimension $\nu(X, L)$ of L . Then the numerical trivial foliation of L is the Kodaira-Iitaka fibration of L .*

Both theorems together imply a sufficient geometric criterion for

$$\kappa(X, L) < \nu(X, L)$$

where L is a pseudo-effective line bundle on X : Suppose that \mathcal{F}_L is the numerically trivial foliation of L and

$$\text{codim}\{\text{leaves of } \mathcal{F}_L\} = \nu(X, L).$$

Then the Kodaira dimension is strictly smaller than the numerical dimension if \mathcal{F}_L is a genuine foliation, i.e. not induced by a fibration.

The converse of this criterion is in general not true: Let X be the product of an elliptic curve C and a smooth projective variety and let L be the pullback of a

line bundle in $\text{Pic}^0(C)$ with infinite order. Obviously, L is nef, $\nu(L) = 0$ and the numerically trivial foliation of L is induced by the mapping of X to a point. On the other hand $\kappa(X, L) < 0$.

Note however that for $L = K_X$ and $\nu(K_X) = 0$ Campana and Peternell [CP04] proved that K_X is effective. Hence an example as above cannot occur for the canonical bundle, and one might hope to prove a converse of the criterion at least for K_X .

In [Eck04a] 2 surface examples are illustrating this criterion. Unfortunately another example [Eck04a, 4.3] dealing with the anticanonical bundle of \mathbb{P}^2 blown up in 9 points lying sufficiently general on a smooth elliptic curve shows that

$$\text{codim}\{\text{leaves of } \mathcal{F}_L\} > \nu(X, L)$$

may also occur. Of course, the generalized Abundance Conjecture together with Theorem 3.2 imply that this is not the case for the canonical bundle.

1. SINGULAR FOLIATIONS

Holomorphic foliations on complex manifolds are usually defined as involutive subbundles of the tangent bundle. Then the classical theorem of Frobenius asserts that through every point there is a unique integral complex submanifold [Miy86]. Singular foliations may be defined as involutive coherent subsheaves of the tangent bundle which are furthermore saturated, that is their quotient with the tangent bundle is torsion free. In points where the rank is maximal one may use again the Frobenius theorem to get leaves.

Later on we use the following notation:

Definition 1.1. *Let X be a complex manifold and $\mathcal{F} \subset T_X$ a saturated involutive subsheaf. Then the analytic subset*

$$\{x \in X : \mathcal{F}/m_{X,x}\mathcal{F} \rightarrow T_{X,x} \text{ is not injective}\}$$

is called the singular locus of \mathcal{F} and is denoted by $\text{Sing } \mathcal{F}$. The dimension of $\mathcal{F}/m_{X,x}\mathcal{F}$ in a point $x \in X - \text{Sing } \mathcal{F}$ is called rank of \mathcal{F} and denoted by $\text{rk}(\mathcal{F})$.

Because \mathcal{F} is saturated we have $\text{codim } \text{Sing } \mathcal{F} \geq 2$. The existence of leaves means that around every point $x \in X - \text{Sing } \mathcal{F}$ there is an (analytically) open subset $U \subset X - \text{Sing } \mathcal{F}$ with coordinates z_1, \dots, z_n , $n = \dim X$, such that the leaves of \mathcal{F} are the fibers of the projection onto the coordinates z_{k+1}, \dots, z_n where $k = \text{rk}(\mathcal{F})$. In particular the leaves have dimension $\text{rk}(\mathcal{F})$.

To construct numerically trivial foliations we need a local description of several operations applied on two foliations. We start with the easiest configuration:

Proposition 1.2. *Let $\mathcal{G} \subset \mathcal{F}$ be two foliations on a complex manifold X , $\text{rk}(\mathcal{F}) = k$, $\text{rk}(\mathcal{G}) = l$, $l < k$. Then for all $x \in X - (\text{Sing } \mathcal{F} \cup \text{Sing } \mathcal{G})$ there is an open neighborhood $U \subset X - (\text{Sing } \mathcal{F} \cup \text{Sing } \mathcal{G})$ with coordinates z_1, \dots, z_n such that the leaves of \mathcal{F} are the fibers of the projection onto the last $n - k$ coordinates and the leaves of \mathcal{G} are the fibers of the projection onto the last $n - l$ coordinates.*

Proof. This is an easy consequence of the theorem on implicitly defined functions. Note that neither $\text{Sing } \mathcal{G}$ need to be contained in $\text{Sing } \mathcal{F}$ nor vice versa. \square

Definition 1.3. *Let \mathcal{F} and \mathcal{G} be two foliations on a complex manifold X . Then $\mathcal{F} \cap \mathcal{G} \subset T_X$ is called the intersection foliation of \mathcal{F} and \mathcal{G} .*

Note that $\mathcal{F} \cap \mathcal{G}$ is certainly involutive but may be not saturated: the rank of $\mathcal{F} \cap \mathcal{G}$ can even jump in codim 1 subsets. To get a better picture in local coordinates we nevertheless think of it as a foliation and denote by $\text{Sing}(\mathcal{F} \cap \mathcal{G})$ the analytic locus where the rank jumps.

Proposition 1.4. *Let \mathcal{F} and \mathcal{G} be two foliations on a complex manifold X with $\text{rk}(\mathcal{F}) = k$, $\text{rk}(\mathcal{G}) = m$ and $\text{rk}(\mathcal{F} \cap \mathcal{G}) = l$. Let $x \in X$ be a point which is not singular for \mathcal{F} , \mathcal{G} and $\mathcal{F} \cap \mathcal{G}$. Then there exists an open neighborhood*

$$U \subset X - (\text{Sing } \mathcal{F} \cup \text{Sing } \mathcal{G} \cup \text{Sing}(\mathcal{F} \cap \mathcal{G}))$$

of x with coordinates z_1, \dots, z_n such that

- (i) the leaves of \mathcal{F} in U are the fibers of the projection on z_{k+1}, \dots, z_n ,
- (ii) the leaves of $\mathcal{F} \cap \mathcal{G}$ in U are the fibers of the projection on z_{l+1}, \dots, z_n and
- (iii) the leaves of \mathcal{G} in U are the fibers of the projection on $z_{l+1}, \dots, z_k, g_{m+k-l+1}(z), \dots, g_n(z)$ where the g 's are analytic functions with $g_{m+k-l+j}(z)|_{U_x} = z_{k+j}$ on

$$U_x = \{z \in U : z_{l+1}(z) = z_{l+1}(x), \dots, z_k(z) = z_k(x)\}.$$

Proof. Again this results from applying the theorem on implicitly defined functions several times. Since the geometry is more difficult than in Prop. 1.2 (see Figure 1) we present more details: Choose coordinates z_1, \dots, z_n for \mathcal{F} and $\mathcal{F} \cap \mathcal{G}$ in a neighborhood

$$V' \subset X - (\text{Sing } \mathcal{F} \cup \text{Sing } \mathcal{G} \cup \text{Sing}(\mathcal{F} \cap \mathcal{G}))$$

of x as in Prop. 1.2. Since the leaves of \mathcal{G} contain the leaves of $\mathcal{F} \cap \mathcal{G}$ we can describe the leaves of \mathcal{G} in V' (possibly restricted) as the fibers of the projection given by analytic functions g_{m+1}, \dots, g_n only depending on z_{l+1}, \dots, z_n . Furthermore we know that for a fixed point (z_{k+1}, \dots, z_n) the fibers of the projection from the leaf of \mathcal{F} given by g_{m+1}, \dots, g_n are leaves of $\mathcal{F} \cap \mathcal{G}$.

Consequently an application of the theorem on implicitly defined functions gives us (after possibly reordering the g 's) coordinates

$$z_1, \dots, z_l, z'_{l+1} = g_{m+1}(z), \dots, z'_k = g_{m+k-l}(z), z_{k+1}, \dots, z_n$$

in an open subset $V'' \subset V'$ such that the leaves of \mathcal{F} resp. $\mathcal{F} \cap \mathcal{G}$ are still the projection onto the last $n - k$ resp. $n - l$ coordinates, and the leaves of \mathcal{G} are the fibers of the projection onto

$$z'_{l+1}, \dots, z'_k, g_{m+k-l+1}(z), \dots, g_n(z).$$

Now we fix $z'_{l+1} = a_{l+1}, \dots, z'_k = a_k$. Using again the theorem on implicitly defined functions we see that (after possibly another reordering of the g 's)

$$\begin{aligned} & z_1, \dots, z_l, z'_{l+1}, \dots, z'_k, \\ & z'_{k+1} = g_{m+k-l+1}(a_{l+1}, \dots, a_k, z_{k+1}, \dots, z_n), \\ & \vdots \\ & z'_{n-m+l} = g_n(a_{l+1}, \dots, a_k, z_{k+1}, \dots, z_n), \\ & z_{n-m+l+1}, \dots, z_n \end{aligned}$$

are coordinates in an open subset $V \subset V''$ having all the properties claimed in the proposition. \square

For our purposes the most important operation on two holomorphic foliations \mathcal{F} and \mathcal{G} on a complex manifold X is the *union* $\mathcal{F} \sqcup \mathcal{G}$. We define it as the foliation given by the smallest saturated involutive subsheaf of T_X containing both \mathcal{F} and \mathcal{G} . Such a sheaf exists because saturated foliations contained in each other have different ranks, the intersection of two foliations is again a foliation and T_X is involutive.

Besides this pure existence statement there is an inductive algebraic construction of $\mathcal{F} \sqcup \mathcal{G}$:

$$\begin{aligned} \mathcal{H}_1 &:= \text{saturation of } \mathcal{F} + \mathcal{G} \\ \mathcal{H}_2 &:= \text{saturation of } \mathcal{H}_1 + [\mathcal{H}_1, \mathcal{H}_1] \\ &\vdots \end{aligned}$$

and so on until $\mathcal{H}_m = \mathcal{H}_{m+1}$ which means $[\mathcal{H}_m, \mathcal{H}_m] \subset \mathcal{H}_m$. Then $\mathcal{H}_m = \mathcal{F} \sqcup \mathcal{G}$. This is a local construction hence for open subsets $U \subset X$ we have

$$\mathcal{F}|_U \sqcup \mathcal{G}|_U = (\mathcal{F} \sqcup \mathcal{G})|_U.$$

We want to describe an inductive geometric construction of $\mathcal{F} \sqcup \mathcal{G}$ on open subsets

$$U \subset X - (\text{Sing } \mathcal{F} \cup \text{Sing } \mathcal{G} \cup \text{Sing } (\mathcal{F} \cap \mathcal{G})) - Z$$

where Z is an analytic subset of $X - (\text{Sing } \mathcal{F} \cup \text{Sing } \mathcal{G} \cup \text{Sing } (\mathcal{F} \cap \mathcal{G}))$. Following the inductive steps of this construction we will later on prove Key Lemma 2.5.

Start with a neighborhood U of a point $x \in X - (\text{Sing } \mathcal{F} \cup \text{Sing } \mathcal{G} \cup \text{Sing } (\mathcal{F} \cap \mathcal{G}))$ having coordinates z_1, \dots, z_n as in Prop. 1.4. Define a foliation \mathcal{G}' on U whose leaves are the fibers of the projection on $z_{l+1}, \dots, z_{n-m+l}$. Figure 1 illustrates that in general $\mathcal{F} + \mathcal{G}' \neq \mathcal{F} \sqcup \mathcal{G}$ (take the fibers of the vertical projection as leaves of \mathcal{F} whereas the leaves of \mathcal{G} are the horizontal lines twisted around in vertical direction): Denoting the projection on z_{k+1}, \dots, z_n by $\pi_{\mathcal{F}}$ we examine instead r -tuples of points x_1, \dots, x_r in fibers $\pi_{\mathcal{F}}^{-1}(y)$ of points $y \in \pi_{\mathcal{F}}(U) \subset \mathbb{C}^{n-k}$. If $T_{\mathcal{G}}(x_i) \subset T_{X, x_i}$ indicates the space of directions tangent to \mathcal{G} in x_i we have a sequence of inclusions

$$0 \subset d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_1)) \subset d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_1)) + d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_2)) \subset \dots \subset \sum_{i=1}^r d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i)) \subset T_{\mathbb{C}^{n-k}, y}.$$

There is an $r \in \mathbb{N}$ and a Zariski open subset of the r -fold product

$$\pi_{\mathcal{F}}^{-1}(y) \times \dots \times \pi_{\mathcal{F}}^{-1}(y)$$

such that

- (i) all inclusions in the above sequence are strict and
- (ii) $d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x')) \subset \sum_{i=1}^r d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i))$ for every point $x' \in \pi_{\mathcal{F}}^{-1}(y)$.

Varying $y \in \pi_{\mathcal{F}}(U)$ may change the number r and the dimensions of the vector spaces

$$\sum_{i=1}^s d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i)), \quad s = 1, \dots, r.$$

But again there is an analytic subset $Z_U \subset \pi_{\mathcal{F}}(U)$ such that for $y \in V := \pi_{\mathcal{F}}(U) - Z_U$ the dimensions and r remain constant. Since everything is defined intrinsically the sets $\pi_{\mathcal{F}}^{-1}(Z_U)$ glue together to an analytic subset Z of $X - (\text{Sing } \mathcal{F} \cup \text{Sing } \mathcal{G} \cup \text{Sing } (\mathcal{F} \cap \mathcal{G}))$. Furthermore we can find r sections $\sigma_i : V' \rightarrow U$ of $\pi_{\mathcal{F}}$ such that

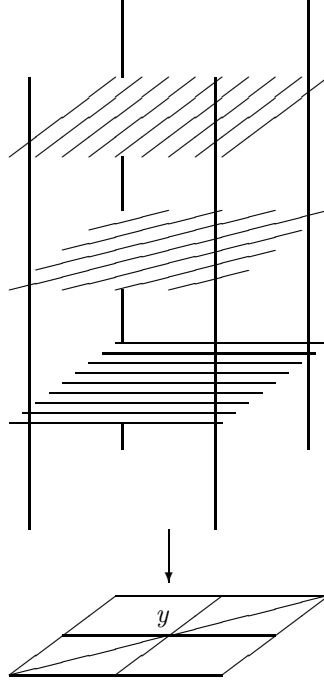


FIGURE 1.1. Union of two foliations

- (i) the points $x_i := \sigma_i(y)$ produce a sequence of tangent subspaces $\sum_{i=1}^s d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i))$, $s = 1, \dots, r$, and
- (ii) if $\pi : U \rightarrow \mathbb{C}^{k-l}$ is the projection onto z_{l+1}, \dots, z_k , the map $\pi \circ \sigma_i$ is constant.

The V' are possibly smaller subsets of V covering V .

To get the announced inductive construction of $\mathcal{F} \sqcup \mathcal{G}$ on $\pi_{\mathcal{F}}^{-1}(V')$ we need another little observation: Since the holomorphic functions g_j defining $\pi_{\mathcal{G}}$ do not depend on z_1, \dots, z_l (see proof of Prop. 1.4) the tangent space

$$d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x))$$

does not change for different x in the intersection of a fixed $\pi_{\mathcal{F}}$ - and a π -fiber. Furthermore the fibers of π consist of leaves of \mathcal{G} .

Now we construct inductively foliations \mathcal{F}_i , $i = 0, \dots, r$ on $\pi_{\mathcal{F}}^{-1}(V')$. We start with

$$\mathcal{F}_0 := \mathcal{F} \cap \pi_{\mathcal{F}}^{-1}(V').$$

Because of the observation above the leaves of \mathcal{G} in $\pi^{-1}(\pi(x_1))$ map onto the leaves of a smooth foliation \mathcal{G}_1 on V' which is induced by a projection $\pi_{\mathcal{G}_1}$. Put

$$\mathcal{F}_1 := \pi_{\mathcal{F}}^{-1}(\mathcal{G}_1)$$

and let $\pi_{\mathcal{F}_1} := \pi_{\mathcal{G}_1} \circ \pi_{\mathcal{F}}$ be the projection whose fibers are the leaves of \mathcal{F}_1 .

The observation and the properties of the x_1, \dots, x_r imply that $T_{\mathcal{G}|_{\pi^{-1}(\pi(x_2))}}$ maps onto an involutive **subbundle** of $T_{\pi_{\mathcal{F}_1}(\pi_{\mathcal{F}}^{-1}(V'))}$ and consequently the leaves of \mathcal{G} in

$\pi^{-1}(\pi(x_2))$ also map onto leaves of a smooth foliation \mathcal{G}_2 on $\pi_{\mathcal{F}_1}(\pi_{\mathcal{F}}^{-1}(V'))$. Define

$$\mathcal{F}_2 := \pi_{\mathcal{F}_1}^{-1}(\mathcal{G}_2)$$

and continue inductively setting

$$\mathcal{F}_i := \pi_{\mathcal{F}_{i-1}}^{-1}(\mathcal{G}_i)$$

where \mathcal{G}_i is the image of the leaves of \mathcal{G} in $\pi^{-1}(\pi(x_i))$ on $\pi_{\mathcal{F}_{i-1}}(\pi_{\mathcal{F}}^{-1}(V'))$.

By construction these foliations \mathcal{F}_s have as tangent space in a point $x \in \pi_{\mathcal{F}}^{-1}(V')$

$$d\pi_{\mathcal{F}}(x)^{-1} \left(\sum_{i=1}^s d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i)) \right)$$

where $\pi_{\mathcal{F}}(x_i) = \pi_{\mathcal{F}}(x)$ for all i . In addition \mathcal{F}_r contains all leaves of \mathcal{F} and \mathcal{G} in $\pi_{\mathcal{F}}^{-1}(V')$: otherwise there is a point $y \in V'$ and a point $x \in \pi_{\mathcal{F}}^{-1}(y)$ such that

$$d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x)) \not\subset \sum_{i=1}^r d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i)),$$

$\pi_{\mathcal{F}}(x_i) = \pi_{\mathcal{F}}(x)$ for all i .

On the other hand $T_{\mathcal{F} \sqcup \mathcal{G}}(x)$ must contain every tangent subspace

$$d\pi_{\mathcal{F}}(x)^{-1}(T_{\mathcal{G}}(x'))$$

of points x' with $\pi_{\mathcal{F}}(x') = \pi_{\mathcal{F}}(x)$ since $\pi_{\mathcal{F}}^{-1}\pi_{\mathcal{F}}(\pi_{\mathcal{G}}^{-1}\pi_{\mathcal{G}}(x'))$ is contained in a leaf of $\mathcal{F} \sqcup \mathcal{G}$. Consequently, $d\pi_{\mathcal{F}}(x)^{-1}(\sum_{i=1}^r d\pi_{\mathcal{F}}(T_{\mathcal{G}}(x_i))) \subset T_{\mathcal{F} \sqcup \mathcal{G}}(x)$ and on $\pi_{\mathcal{F}}^{-1}(V')$ we have

$$\mathcal{F} \sqcup \mathcal{G} = \mathcal{F}_r.$$

An important type of foliations are those induced in a unique way by meromorphic maps $f : X \dashrightarrow Y$ from a compact complex manifold X to another complex manifold Y : Take the relative tangent sheaf of f on the Zariski open subset U where f is smooth and saturate.

Proposition 1.5. *Let X be a compact complex manifold and $f : X \dashrightarrow Y_1$, $g : X \dashrightarrow Y_2$ two meromorphic maps with induced foliations \mathcal{F} and \mathcal{G} on X . Then $\mathcal{F} \sqcup \mathcal{G}$ is also induced by a meromorphic map $h : X \dashrightarrow Z$.*

Proof. Let $\Gamma_f \subset Y_1 \times X$, $\Gamma_g \subset X \times Y_2$ be the graphs of f and g . Consider the product $Y_1 \times X \times X \times Y_2$ and its projections p_1, p_2, p_3, p_4 onto the subsequent factors. A general point (y_1, x_1, x_2, y_2) of the intersection

$$(p_1 \times p_2)^{-1}(\Gamma_f) \cap (p_2 \times p_3 \times p_4)^{-1}(\Gamma_g \times_{Y_2} \Gamma_g) \subset Y_1 \times X \times X \times Y_2$$

satisfies $x_1 \in f^{-1}(y_1)$, $x_1, x_2 \in g^{-1}(y_2)$. Since y_1 and y_2 are uniquely determined by a general point $x_1 \in X$ there is a unique irreducible component $W \subset Y_1 \times X \times X \times Y_2$ in this intersection projecting surjectively on X via p_2 such that the fiber over a general point $x \in X$ is a unique g -fiber. $(p_1 \times p_3)(W) \subset Y_1 \times X$ is a family of compact complex subspaces of X parametrized by Y_1 and covering X . Hence there is a meromorphic map from Y_1 to the Douady space of X and we call the image of this map Z_1 . The image $W_1 \subset Z_1 \times X$ of $(p_1 \times p_3)(W)$ in the universal family over the Douady space of X has the following properties:

- (i) Every fiber of W_1 over Z_1 consists of g -fibers.
- (ii) For two points in the same fiber of W_1 over Z_1 there is a sequence of f - and g -fibers connecting them.

- (iii) Given an analytic subset $Z \subset X$ two general points in a general fiber may be connected by a sequence of f - and g -fibers such that two subsequent fibers do not intersect in Z .

Points connected by a sequence of f - and g -fibers satisfying (iii) with

$$Z = \text{Sing}(\mathcal{F} \sqcup \mathcal{G}) \cup \{\text{indeterminacy loci of } f \text{ and } g\}$$

must lie in the same leaf of $\mathcal{F} \sqcup \mathcal{G}$.

Repeat the construction from above but interchange the rôles of f and g : Take the generic irreducible component of the intersection

$$(p_1 \times p_2 \times p_3)^{-1}(\Gamma_f \times_{Y_1} \Gamma_f) \cap (p_3 \times p_4)^{-1}(\Gamma_g) \subset Y_1 \times X \times X \times Y_2$$

under the projection p_3 , project it via $p_2 \times p_4$ in $X \times Z_1$ and map it via the universal properties of the Douady space into a family $W_2 \subset Z_2 \times X$ of compact complex subspaces covering X . Now every fiber of W_2 consists of f -fibers and W_2 also satisfies (ii) and (iii).

Continue this construction interchanging the rôles of f and g in each step until the fiber dimension of W_k over Z_k does not rise in the next step. This is the case iff for every fiber F and every point $x \in F$ there is an f - and g -fiber through x contained in F . Through a general point x these fibers are unique. Since W_k also satisfies (iii) two fibers containing the same general point x must be equal. Hence $W_k \subset Z_k \times X \rightarrow X$ is generically 1:1 and defines a meromorphic map $h : X \dashrightarrow Z_k$ whose induced foliation \mathcal{H} contains \mathcal{F} and \mathcal{G} . On the other hand the fibers of W_k are contained in leaves of $\mathcal{F} \sqcup \mathcal{G}$ by construction. Taken all together we have shown:

$$\mathcal{H} = \mathcal{F} \sqcup \mathcal{G}.$$

□

Remark 1.6. This construction closely resembles that of Campana's reduction map [Cam81, Cam94]. The difference is that in the above construction for any given analytic subset $Z \subset X$ two general points lie in the same fiber iff they can be connected without touching Z . The easiest example where the two reduction maps fall apart are the quotients with respect to the pencil of lines through a point in \mathbb{P}^2 .

We generalize this construction to the following

Definition 1.7. Let X be a compact complex manifold. A covering family $(C_t)_{t \in T}$ of complex subspaces of X is called *generically connecting* iff for any analytic subset $Z \subset X$ two general points are connected by a finite sequence of elements in (C_t) such that two subsequent elements do not intersect in Z .

A meromorphic map $f : X \dashrightarrow Y$ is called the *generic reduction map with respect to $(C_t)_{t \in T}$* iff the general fibers are generically C_t -connected and every element of (C_t) is contained in a fiber. Here, fibers of f are defined via the graph of f .

The construction described above shows that for every family $(C_t)_{t \in T}$ there exists a unique generic reduction map.

Another difference between the generic and Campana's reduction map is the stability under modifications: Let X be a compact complex manifold and $(C_t)_{t \in T}$ a covering family of complex subspaces of X . Let $f : X \dashrightarrow Y$ be the generic quotient and $g : X \dashrightarrow Z$ Campana's quotient with respect to (C_t) . If $\pi : \widehat{X} \rightarrow X$ is a modification of compact Kähler manifolds then the generic quotient of \widehat{X} w.r.t. the

strict or total transforms of (C_t) is described by $f \circ \pi$ whereas in general Campana's quotient is described by $g \circ \pi$ only w.r.t. the total transforms of (C_t) (cf. the pencil of lines through a point in \mathbb{P}^2).

2. NUMERICALLY TRIVIAL FOLIATIONS

From now on let X be a compact Kähler manifold. On $X - \text{Sing } \mathcal{F}$ a foliation \mathcal{F} is described by a subbundle $T_{\mathcal{F}}$ of the tangent bundle T_X . Then $(T_X/T_{\mathcal{F}})^*$ is a (holomorphic) subbundle of $T_X^* = \Omega_X^1$. Hence the subbundle generates (p, p) -forms on $X - \text{Sing } \mathcal{F}$ which we collect in the set

$$\mathcal{E}^{p,p}(X, \mathcal{F}) \subset \mathcal{E}^{p,p}(X - \text{Sing } \mathcal{F}).$$

Note that $\mathcal{E}^{p,p}(X, \mathcal{F}) = 0$ for $p > \dim X - \text{rk } \mathcal{F}$.

An $(n-p, n-p)$ -form $u \in \mathcal{D}^{(n-p, n-p)}(X - \text{Sing } \mathcal{F})$ is called **test form** for \mathcal{F} iff for all $v \in \mathcal{E}^{p,p}(X, \mathcal{F})$

$$v \wedge u = 0.$$

Definition 2.1. Let X be a compact Kähler manifold with Kähler form ω and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudo-effective $(1, 1)$ -class. A foliation \mathcal{F} is called *numerically trivial* iff for all $1 \leq p \leq n-1$ and for all test forms $u \in \mathcal{D}^{(n-p, n-p)}(X - \text{Sing } \mathcal{F})$

$$(NT)_u \quad \lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_X |(T_{ac} + \epsilon\omega)^p \wedge u| = 0$$

where the T 's run through all currents with analytic singularities representing α with $T \geq -\epsilon\omega$ and T_{ac} is the absolute continuous part of T in the Lebesgue decomposition.

Note that $|(T_{ac} + \epsilon\omega)^p \wedge u|$ is the total variation of the measure $(T_{ac} + \epsilon\omega)^p \wedge u$. Since all occurring currents are absolutely continuous they may be written as forms with (at least) L^1_{loc} functions as coefficients. In particular $(T_{ac} + \epsilon\omega)^p \wedge u = f \cdot \omega^n$ for an L^1_{loc} function f and

$$|(T_{ac} + \epsilon\omega)^p \wedge u| = |f| \cdot \omega^n.$$

The usual facts about absolute values like the triangle inequality follow immediately from this formula.

To verify numerical triviality of a foliation \mathcal{F} we only need to check condition $(NT)_u$ for special test forms for \mathcal{F} :

Definition 2.2. Let \mathcal{F} be a foliation of rank k on a complex n -dimensional manifold X and $U \subset X - \text{Sing } \mathcal{F}$ an open subset with coordinates z_1, \dots, z_n such that the leaves of \mathcal{F} in U are the fibers of the projection onto the last $n-k$ coordinates.

Let $1 \leq p \leq n-1$. An $(n-p, n-p)$ form $\sum_{|I|=|J|=n-p} a_{IJ} dz_I \wedge d\bar{z}_J$ on U is called *constant test form* for \mathcal{F} on U iff the $a_{IJ} \in \mathbb{C}$ are constant and

$$\#(I \cap \{k+1, \dots, n\}) \leq n-k-p \text{ and } \#(J \cap \{k+1, \dots, n\}) \leq n-k-p \implies a_{IJ} = 0.$$

Theorem 2.3. Let X be a compact Kähler manifold with Kähler form ω and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudoeffective $(1, 1)$ class. A foliation \mathcal{F} on X of $\text{rk } k$ is *numerically trivial* w.r.t. α iff there exists a covering $\{U_i\}$ of relatively compact open subsets of $X - \text{Sing } \mathcal{F}$ satisfying the following conditions: On each U_i there are coordinates z_1, \dots, z_n such that

- (i) the leaves of \mathcal{F} are the fibers of the projection onto the last $n-k$ coordinates,
- (ii) for all $1 \leq p \leq n-1$ and for all real constant test forms u for \mathcal{F} in U_i the equality $(NT)_u$ is true.

Proof. In each point $x \in U_i$ constant test forms for \mathcal{F} in U_i generate all forms of $\mathcal{E}^{p,p}(X, \mathcal{F})$. Hence arbitrary test forms for \mathcal{F} in U_i can be approximated by locally constant test forms: Let u be a real $(n-p, n-p)$ test form for \mathcal{F} in U_i . Then for every $\epsilon > 0$ there exists a locally constant test form u_ϵ such that

$$-\epsilon\omega^{n-p} < u - u_\epsilon < \epsilon\omega^{n-p}$$

in U_i . Consequently we get

$$\begin{aligned} \int_{U_i} |(T_{ac} + \epsilon\omega)^p \wedge u| &\leq \int_{U_i} |(T_{ac} + \epsilon\omega)^p \wedge (u - u_\epsilon)| + \int_{U_i} |(T_{ac} + \epsilon\omega)^p \wedge u_\epsilon| \\ &\leq \epsilon \int_{U_i} (T_{ac} + \epsilon\omega)^p \wedge \omega^{n-p} + \int_{U_i} |(T_{ac} + \epsilon\omega)^p \wedge u_\epsilon| \end{aligned}$$

for every current $T \in \alpha[-\epsilon\omega]$ with analytic singularities. By Boucksom's theory of moving intersection numbers [Bou02][3.2] we have

$$\limsup_{\epsilon \downarrow 0} \int_{U_i} (T_{ac} + \epsilon\omega)^p \wedge \omega^{n-p} \leq \limsup_{\epsilon \downarrow 0} \int_X (T_{ac} + \epsilon\omega)^p \wedge \omega^{n-p} = (\alpha \cdot \omega^{n-p})_{\geq 0}.$$

For the second summand note that u_ϵ is a linear combination of globally constant test forms for \mathcal{F} in U_i multiplied with characteristic functions. Using the assumption we conclude

$$\limsup_{\epsilon \downarrow 0} \int_{U_i} |(T_{ac} + \epsilon\omega)^p \wedge u_\epsilon| = 0.$$

□

To justify the definition of numerical triviality we prove

Lemma 2.4. *Let $U \subset \mathbb{C}^n$ be an open subset and $\pi : U \rightarrow \mathbb{C}^{n-k}$ the projection onto the last $n-k$ factors. If T is a closed absolutely continuous $(1,1)$ -current on U such that*

$$\int_U |T \wedge u| = 0$$

for all test forms $u \in \mathcal{D}^{n-1, n-1}(U)$ for π then there exists a $(1,1)$ -current S on $\pi(U) \subset \mathbb{C}^{n-k}$ such that

$$T \equiv \pi^* S.$$

Proof. Since T is absolutely continuous there are L^1_{loc} -functions f_{ij} such that

$$T \equiv \sum f_{ij} dz_i \wedge d\bar{z}_j.$$

Hence $\int_U |T \wedge u| = 0$ for all π -test forms $u \in \mathcal{D}^{n-1, n-1}(U)$ tells us

$$(i, j) \notin \{k+1, \dots, n\} \times \{k+1, \dots, n\} \implies f_{ij} \equiv 0.$$

But then the closedness of T implies that for other f_{ij} 's the partial derivatives $\frac{\partial}{\partial z_l} f_{ij}$ and $\frac{\partial}{\partial \bar{z}_l} f_{ij}$ vanish (in the sense of currents) if $l \notin \{k+1, \dots, n\}$. Consequently these f_{ij} 's do not depend on z_l , $l = 1, \dots, k$, and

$$T \equiv \sum_{i, j=k+1}^n f_{ij} dz_i \wedge d\bar{z}_j$$

may be interpreted as pulled back from a current S on $\pi(U)$. □

We want to show that there is always a *maximal* numerically trivial foliation \mathcal{F} for a pseudoeffective $(1,1)$ -class α meaning that every foliation numerically trivial w.r.t. α is contained in \mathcal{F} . This will be a direct consequence of

Theorem 2.5 (Key Lemma). *Let X be a compact Kähler manifold with Kähler form ω and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudoeffective $(1,1)$ class. If \mathcal{F} and \mathcal{G} are numerically trivial foliations then $\mathcal{F} \sqcup \mathcal{G}$ will be numerically trivial, too.*

The proof is divided in two main parts: First we show in Thm. 2.6 that we can neglect arbitrary (small neighborhoods of) analytic subsets when checking $(NT)_u$. This is done by applying Boucksom's uniform bounds for the occurring integrals on generalized Lelong numbers.

The second part follows the inductive construction of $\mathcal{F} \sqcup \mathcal{G}$ on open subsets U which do not intersect an arbitrarily small neighborhood W of $\text{Sing } \mathcal{F} \cup \text{Sing } \mathcal{G} \cup \text{Sing } (\mathcal{F} \cap \mathcal{G})$ and the subset Z analytic in $X - W$ on which the construction is not possible. If z_1, \dots, z_n are holomorphic coordinates on U such that $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are defined as in Prop. 1.4 we prove first that the foliation \mathcal{G}' defined on U by projection onto $z_{l+1}, \dots, z_{n-m+l}$ is numerically trivial (Prop. 2.13). Next we show that $\mathcal{F} + \mathcal{G}'$ is numerically trivial in U (Prop. 2.7; this is the first step of the inductive construction) and continue until we reach $\mathcal{F} \sqcup \mathcal{G}$. We begin with

Theorem 2.6. *Let X be an n -dimensional compact Kähler manifold with Kähler form ω and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudoeffective $(1,1)$ class. Let $U \subset X$ be an open subset with coordinates z_1, \dots, z_n and $L = \{z_1 = \dots = z_l = 0\} \subset U$ a linear subspace. Then for every exhaustion $K_i \subset \subset K_{i+1} \subset \subset X$ of $X - L = \bigcup K_i$ we have for all $1 \leq p \leq n - 1$*

$$\lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_{(X - K_i) \cap U} (T_{ac} + \epsilon\omega)^p \wedge \omega^{n-p} \xrightarrow{i \rightarrow \infty} 0$$

where the T 's run through all currents with analytic singularities representing α and $T \geq -\epsilon\omega$.

Proof. Let us consider the generalized Lelong numbers of currents $(T_{ac} + \epsilon\omega)^p$ in points $x = (0, \dots, 0, x_{l+1}, \dots, x_n) \in L$ with respect to the plurisubharmonic weight

$$\phi_x(z) = \log\left(\sum_{k=1}^l |z_k - x_k|^2 + \sum_{k=l+1}^n |z_k - x_k|^q\right)$$

where q is some integer $\geq n - l$. The advantage of this weight is that for a given $r > 0$ the number of subsets $\{z : |\phi_x(z)| \leq \log r\}$ necessary to cover L is $\leq C \cdot r^{-\frac{2}{q} \dim L}$ for some constant $C > 0$ independent of r . Furthermore there are two constants $C_1, C_2 > 0$ such that

$$C_1 \omega \leq \frac{i}{2} \partial \bar{\partial} e^{\phi_x} \leq C_2 \omega.$$

By definition the generalized Lelong number $\nu((T_{ac} + \epsilon\omega)^p, \phi_x)$ is the decreasing limit for $t \rightarrow -\infty$ of

$$\begin{aligned} \nu((T_{ac} + \epsilon\omega)^p, \phi_x, t) &= \int_{\phi_x(z) < t} (T_{ac} + \epsilon\omega)^p \wedge (\partial \bar{\partial} \phi_x)^{n-p} \\ &= \frac{1}{(\pi e^{2t})^{n-p}} \int_{\phi_x(z) < t} (T_{ac} + \epsilon\omega)^p \wedge (\partial \bar{\partial} e^{\phi_x})^{n-p} \end{aligned}$$

where the second equality follows from a formula proven in [Dem00, (2.13)].

Set $t = \log r$. On the one hand, for $r \leq r_0$ one has

$$\nu((T_{ac} + \epsilon\omega)^p, \phi_x, \log r) \leq \nu((T_{ac} + \epsilon\omega)^p, \phi_x, \log r_0) \leq \frac{C_2}{(\pi r_0^2)^{n-p}} \int_X (T_{ac} + \epsilon\omega)^p \wedge \omega^{n-p}.$$

But T has analytic singularities. Hence using an idea of Boucksom [Bou02][3.1.12] $\int_X (T_{ac} + \epsilon\omega)^p \wedge \omega^{n-p}$ is bounded from above by a constant depending only on the cohomology class α of T . On the other hand,

$$(\pi r^2)^{n-p} \nu((T_{ac} + \epsilon\omega)^p, \phi_x, \log r) \geq C_1 \int_{e^{\phi_x(z)} < r} (T_{ac} + \epsilon\omega)^p \wedge \omega^{n-p}.$$

The claim follows from the choice of l in the definition of ϕ_x and the upper bound on the number of level subsets covering L . \square

From now on let X be an n -dimensional compact Kähler manifold with Kähler form ω and $\alpha \in H^{1,1}(X, \mathbb{R})$ a pseudo-effective $(1, 1)$ -class. We start the second part with the easiest configuration of two foliations \mathcal{F} and \mathcal{G} of X in an open subset $U \subset X$:

Proposition 2.7. *Let z_1, \dots, z_n be coordinates on U such that \mathcal{F} is induced by the projection on z_{k+1}, \dots, z_n and \mathcal{G} by the projection on $z_{l+1}, \dots, z_{n-m+l}$. If \mathcal{F} and \mathcal{G} are numerically trivial on U the foliation $\mathcal{F} + \mathcal{G}$ induced by the projection on $z_{k+1}, \dots, z_{n-m+l}$ is numerically trivial, too.*

Proof. By Theorem 2.3 it is enough to show that constant $(n-p, n-p)$ test forms for $\mathcal{F} + \mathcal{G}$ are \mathbb{C} -linear combinations of test forms for \mathcal{F} and \mathcal{G} . But constant $(n-p, n-p)$ test forms for $\mathcal{F} + \mathcal{G}$ are \mathbb{C} -linear combinations of decomposable $(n-p, n-p)$ forms $dz_I \wedge d\bar{z}_J$ with

$$\begin{aligned} |I \cap \{k+1, \dots, n-m+l\}| &> n-m-k+l-p \text{ or} \\ |J \cap \{k+1, \dots, n-m+l\}| &> n-m-k+l-p. \end{aligned}$$

Assume w.l.o.g. that the inequality for I is satisfied and set

$$q := |I \cap \{k+1, \dots, n-m+l\}|.$$

If $dz_I \wedge d\bar{z}_J$ is not a test form for \mathcal{F} and \mathcal{G} we have

$$\begin{aligned} |I \cap \{k+1, \dots, n\}| &\leq n-k-p \text{ and} \\ |I \cap \{l+1, \dots, n-m+l\}| &\leq n-m-p. \end{aligned}$$

This implies

$$\begin{aligned} |I \cap \{l+1, \dots, k\}| &\leq n-m-p-q \text{ and} \\ |I \cap \{n-m+l+1, \dots, n\}| &> n-k-p-q. \end{aligned}$$

Consequently

$$\begin{aligned} |I| &= |I \cap \{1, \dots, l\}| + |I \cap \{l+1, \dots, k\}| + q + |I \cap \{n-m+l+1, \dots, n\}| \\ &\leq l + n-m-p-q + q + n-k-p-q = n-p + (n-m-k+l-p) - q \\ &< n-p \end{aligned}$$

by the properties of p . This is a contradiction. \square

Next we state a little fact that is useful later on again and again:

Lemma 2.8. *Let z_1, \dots, z_n be coordinates on U such that \mathcal{F} is induced by the projection on z_{k+1}, \dots, z_n and \mathcal{G} by the projection on $z_{l+1}, \dots, z_{n-m+l}$. If $dz_I \wedge d\bar{z}_J$ is not a (constant) $(n-p, n-p)$ test form for \mathcal{F} we have*

$$\{l+1, \dots, k\} \subset I, J.$$

Proof. If $dz_I \wedge d\bar{z}_J$ is not a test form for \mathcal{F} it follows that

$$|I \cap \{k+1, \dots, n\}|, |J \cap \{k+1, \dots, n\}| \leq n - k - p.$$

But then $|I| = |J| = n - p$ implies

$$|I \cap \{1, \dots, k\}| = k.$$

□

From now on fix coordinates z_1, \dots, z_n on U such that the foliations \mathcal{F} and \mathcal{G} are described as in Proposition 1.4.

The main idea for proving the numerical triviality of \mathcal{G}' is to compare the evaluation of test forms on fibers of the projection $\pi : U \rightarrow \mathbb{C}^{k-l}$ onto z_{l+1}, \dots, z_k . Intuitively the numerical triviality should imply that the difference of these values vanishes. Since we always compute integrals on U we first need a comparison lemma for nearby fibers:

Lemma 2.9. *Suppose that $0 \in U \subset \mathbb{C}^n$. Let*

$$\Phi : U \rightarrow \mathbb{C}^n, (z_1, \dots, z_n) \mapsto (z_1, \dots, z_k, z'_{k+1} = \Phi_{k+1}(z), \dots, z'_n = \Phi_n(z))$$

be a coordinate transformation such that

$$\Phi_{k+j}(z)|_{U_0} = z_{k+j} \text{ on } U_0 = \{z \in U : z_{l+1} = \dots = z_k = 0\}.$$

Set $U_\delta := \pi^{-1}(B_\delta(0))$ and note that $\Phi(U_\delta) = U_\delta$. Then for every real $(n-p, n-p)$ -form u with $p \leq n-k+l$ which is not a test form for \mathcal{F} there exists a constant $C > 0$ independent of δ such that for every δ small enough the inequality of $(n-p, n-p)$ -forms

$$-C\delta\omega_{|\pi^{-1}(x)}^{n-p-k+l} < (\Phi_*(u) - u)|_{\pi^{-1}(x)} < C\delta\omega_{|\pi^{-1}(x)}^{n-p-k+l}$$

is true on all fibers $\pi^{-1}(x)$, $x \in B_\delta(0)$.

Proof. We can replace ω by the standard $(1, 1)$ -form $\sum dz_i \wedge d\bar{z}_i$ on \mathbb{C}^n . Let f_{IJ} be the coefficient functions of $(\Phi_*(u) - u)|_{\pi^{-1}(x)}$ w.r.t. the base $dz_I \wedge d\bar{z}_J$. Then $(\Phi_*(u) - u)|_{\pi^{-1}(0)} = 0$ implies $f_{IJ}|_{\pi^{-1}(0)} \equiv 0$. Let $v \in \mathbb{C}^n$ be a direction vector with 0 entries in all coordinates but z_{l+1}, \dots, z_k and $\|v\| = 1$. The mean value theorem gives us for all $x_0 \in \pi^{-1}(0)$

$$\|f_{IJ}(x_0 + t_0v)\| = \left| \frac{d}{dt} f_{IJ}(x_0 + tv) \cdot t_0v \right| = |t_0| \cdot \|D_{(l+1, \dots, k)} f_{IJ}(x_0 + tv) \cdot v\|$$

where the matrices $D_{(l+1, \dots, k)} f_{IJ}$ collect the partial derivatives w.r.t. $x_{l+1}, y_{l+1}, \dots, x_k, y_k$ ($z_j = x_j + iy_j$). These matrices can be considered as continuous families of linear maps and hence their norms are bounded from above by a constant C' . Consequently

$$\|(\Phi_*(u) - u)|_{\pi^{-1}(x)}\|_{\sup} \leq C'\delta$$

and the claim follows. □

Remark 2.10. $\omega|_{\pi^{-1}(x)}$ is the usual restricted form on the submanifold $\pi^{-1}(x)$. But $(\Phi_*(u) - u)|_{\pi^{-1}(x)}$ must be defined as the $(n-p-k+l, n-p-k+l)$ -form obtained from $\Phi_*(u) - u$ by replacing $dz_I \wedge d\bar{z}_J$ with $dz_{I-\{l+1, \dots, k\}} \wedge d\bar{z}_{J-\{l+1, \dots, k\}}$. This makes sense by Lemma 2.8 because u is not a test form for \mathcal{F} .

Proposition 2.11. *Let x be any point in U . If u is a real constant $(n-p, n-p)$ test form for \mathcal{G}' in U but not for \mathcal{F} and $p \leq n-k+l$ then*

$$\lim_{\delta \rightarrow 0} \frac{1}{\text{Vol}(U_\delta)} \lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_{U_\delta} |(T_{ac} + \epsilon\omega)^p \wedge u| = 0$$

where $U_\delta \subset U$ denotes the open subset $\pi^{-1}(B_\delta(\pi(x)))$.

Proof. Using the notation of Proposition 1.4 the map

$$\Phi : U \rightarrow \mathbb{C}^n, (z_1, \dots, z_n) \mapsto (z_1, \dots, z_k, g_{m+k-l+1}(z), \dots, g_n(z), z_{n-m+l+1}, \dots, z_n)$$

describes a coordinate transformation such that $u' := \Phi_*(u)$ is a real constant test form for \mathcal{G} . Hence

$$(*) \quad \lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_U |(T_{ac} + \epsilon\omega)^p \wedge u'| = 0.$$

We replace U by U_δ and want to compare the growth of this limit with that of $\text{Vol}(U_\delta)$. To this purpose it is enough to look at sequences of currents $T_k \in \alpha[-\epsilon_k\omega]$ with analytic singularities and $\epsilon_k \xrightarrow{k \rightarrow \infty} 0$.

Since u is not a test form for \mathcal{F} Lemma 2.8 tells us that u only contains forms $dz_I \wedge d\bar{z}_J$ with $\{l+1, \dots, k\} \subset I, J$. The same is true for $u' = \Phi_*(u)$ in the new coordinates. Hence using Fubini's theorem

$$\int_U |((T_k)_{ac} + \epsilon_k\omega)^p \wedge u'| = \int_{\pi(U)} d\lambda_{\pi(U)}(x) \left(\int_{\pi^{-1}(x)} |((T_k)_{ac} + \epsilon_k\omega)_{|\pi^{-1}(x)}^p \wedge u''| \right)$$

for the $(n-k+l-p, n-k+l-p)$ -form u'' obtained from u as described in the remark above.

Define L^1 -functions

$$f_k : \pi(U) \rightarrow \mathbb{R}^+, x \mapsto \int_{\pi^{-1}(x)} |((T_k)_{ac} + \epsilon_k\omega)_{|\pi^{-1}(x)}^p \wedge u''|.$$

The f_k 's tend to 0 in L^1 -norm, by (*). Convoluting the f_k 's with $\rho_\delta := \frac{1}{\text{Vol}(U_\delta)} \chi_{U_\delta}$ we get

$$\frac{1}{\text{Vol}(U_\delta)} \cdot \lim_{k \rightarrow \infty} \int_{U_\delta} |((T_k)_{ac} + \epsilon_k\omega)^p \wedge u'| = \lim_{k \rightarrow \infty} (\rho_\delta * f_k)(0) = 0$$

because convolution with characteristic functions of open subsets changes L^1 -convergence in sup-norm convergence. We conclude

$$\begin{aligned} & \frac{1}{\text{Vol}(U_\delta)} \limsup_{\epsilon \downarrow 0} \sup_T \int_{U_\delta} |(T_{ac} + \epsilon\omega)^p \wedge u| \leq \\ & \leq \frac{1}{\text{Vol}(U_\delta)} \left(\limsup_{\epsilon \downarrow 0} \sup_T \int_{U_\delta} |(T_{ac} + \epsilon\omega)^p \wedge (u - u')| + \limsup_{\epsilon \downarrow 0} \sup_T \int_{U_\delta} |(T_{ac} + \epsilon\omega)^p \wedge u'| \right) \\ & \leq \frac{1}{\text{Vol}(U_\delta)} \limsup_{\epsilon \downarrow 0} \sup_T \int_{B_\delta(0)} d\lambda_{B_\delta(0)}(x) \left(\int_{\pi^{-1}(x)} |((T_{ac} + \epsilon\omega)_{|\pi^{-1}(x)}^p \wedge (u - u')_{|\pi^{-1}(x)})| \right) \end{aligned}$$

Using Lemma 2.9 we continue this inequality chain with

$$\begin{aligned}
&\leq \frac{C}{\text{Vol}(U_\delta)} \limsup_{\epsilon \downarrow 0} \int_{B_\delta(0)} d\lambda_{B_\delta(0)}(x) \left(\int_{\pi^{-1}(x)} (T_{ac} + \epsilon\omega)_{|\pi^{-1}(x)}^p \wedge \delta\omega_{|\pi^{-1}(x)}^{n-p-k+l} \right) \\
&\leq \frac{C \cdot \delta}{\text{Vol}(U_\delta)} \limsup_{\epsilon \downarrow 0} \int_{U_\delta} (T_{ac} + \epsilon\omega)^p \wedge \omega^{n-p} \\
&\leq \frac{C' \cdot \delta}{\delta^{2(k-l)}} \limsup_{\epsilon \downarrow 0} \delta^{2(n-p)} \cdot \nu((T_{ac} + \epsilon\omega)^p, \log|z|, \log\delta) \\
&\leq C' \cdot \delta \cdot \limsup_{\epsilon \downarrow 0} \int_X (T_{ac} + \epsilon\omega)^p \wedge \omega^{n-p}
\end{aligned}$$

by the estimates in the proof of Theorem 2.6 and $p \leq n - k + l$. This last term tends to 0 if $\delta \rightarrow 0$ because of the uniform bounds of Boucksom. \square

Now we compare different fibers of π :

Proposition 2.12. *Assume that $x = 0$ and let u be again a real $(n-p, n-p)$ form, $p \leq n - k + l$, which is not a test form for \mathcal{F} . Let \bar{y} be another point in $\pi(U) \subset \mathbb{C}^{k-l}$ and $U_{\delta, \bar{y}} := \pi^{-1}(B_\delta(\bar{y}))$. Then*

$$\int_{U_{\delta, \bar{y}}} |(T_{ac} + \epsilon\omega)^p \wedge u| = \int_{U_\delta} |(T_{ac} + \epsilon\omega)^p \wedge u|.$$

Proof. Let $\bar{x} = (z_{l+1}, \dots, z_k) = (x_{l+1}, y_{l+1}, \dots, x_k, y_k) \in B_\delta(0)$. Then $\pi^{-1}(\bar{x})$ and $\pi^{-1}(\bar{x} + \bar{y})$ may be seen as part of the boundary of a ‘‘cylinder’’ $Z(\bar{x})$ obtained by connecting all pairs of points $x' \in \pi^{-1}(\bar{x})$ and $x' + (0, \dots, 0, \bar{y}, 0, \dots, 0) \in \pi^{-1}(\bar{x} + \bar{y})$ with a real line.

Since T_{ac} only has analytic singularities of codimension ≥ 2 the restriction of T_{ac} to these real lines is a smooth form for almost all $x' \in \pi^{-1}(\bar{x})$.

To prove the proposition it is enough to look at u 's which are decomposable forms $dx_{I_0} \wedge dy_{J_0}$ in the real coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. Then

$$\begin{aligned}
(T_{ac} + \epsilon\omega)^p \wedge dx_{I_0} \wedge dy_{J_0} &= \left(\sum_{|I|=|J|=p} T_{IJ} dx_I \wedge dy_J \right) \wedge dx_{I_0} \wedge dy_{J_0} \\
&= T_{I'_0 J'_0} dx_{I'_0} \wedge dy_{J'_0} \wedge dx_{I_0} \wedge dy_{J_0}
\end{aligned}$$

where $I'_0 \cup I_0 = J'_0 \cup J_0 = \{1, \dots, n\}$ and $T_{I'_0 J'_0}$ is a real L_{loc}^1 -function. Let \bar{u} be the decomposable form for which

$$u = \bar{u} \wedge x_{l+1} \wedge y_{l+1} \wedge \dots \wedge x_k \wedge y_k.$$

The form \bar{u} exists because u is not a test form for \mathcal{F} . Then

$$(T_{ac} + \epsilon\omega)^p \wedge \bar{u} = T_{I'_0 J'_0} dx_I \wedge dy_J, \quad I = J = \{1, \dots, l, k+1, \dots, n\}.$$

Since T_{ac} , ω and \bar{u} (as a constant form) are closed forms resp. currents we get

$$d[(T_{ac} + \epsilon\omega)^p \wedge \bar{u}] = 0.$$

If we restrict $(T_{ac} + \epsilon\omega)^p \wedge \bar{u}$ to the cylinder $Z(\bar{x})$ the current remains closed. But in $Z(\bar{x})$ this just means that the derivation in direction of $(0, \dots, 0, \bar{y}, 0, \dots, 0)$ vanishes. This implies that $T_{I'_0 J'_0}$ remains constant on the real lines in $Z(\bar{x})$ where

T_{ac} is smooth. Consequently integrating $|(T_{ac} + \epsilon\omega)^p \wedge \bar{u}|$ on the top $\pi^{-1}(\bar{x} + \bar{y})$ and the bottom $\pi^{-1}(\bar{x})$ of the cylinder $Z(\bar{x})$ gives the same result. We finally calculate:

$$\begin{aligned} & \int_{U_{\delta, \bar{y}}} |(T_{ac} + \epsilon\omega)^p \wedge u| - \int_{U_{\delta}} |(T_{ac} + \epsilon\omega)^p \wedge u| = \\ & = \int_{B_{\delta}(0)} d\bar{x} \left(\int_{\pi^{-1}(\bar{x} + \bar{y})} |(T_{ac} + \epsilon\omega)^p \wedge \bar{u}| - \int_{\pi^{-1}(\bar{x})} |(T_{ac} + \epsilon\omega)^p \wedge \bar{u}| \right) = 0. \end{aligned}$$

□

Now we can show

Proposition 2.13. *If \mathcal{F} and \mathcal{G} are numerically trivial in U w.r.t. α then the foliation \mathcal{G}' will be numerically trivial in U w.r.t. α , too.*

Proof. We have to show that

$$\lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_U |(T_{ac} + \epsilon\omega)^p \wedge u| = 0$$

for every constant $(n-p, n-p)$ -test form for \mathcal{G}' . Since \mathcal{F} is numerically trivial in U we only need to check forms which are not test forms for \mathcal{F} . To this purpose we cover U with open subsets $U_{\delta}^{(i)} := U_{\delta, \bar{y}_i}$ as in the previous proposition. There exists a constant $C > 0$ independent of δ such that we only need $\frac{C}{\text{Vol}(U_{\delta})}$ of these covering sets for every $\delta > 0$. Now we calculate:

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_U |(T_{ac} + \epsilon\omega)^p \wedge u| & \leq \lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \sum_i \int_{U_{\delta}^{(i)}} |(T_{ac} + \epsilon\omega)^p \wedge u| \\ & \leq \sum_i \lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_{U_{\delta}^{(i)}} |(T_{ac} + \epsilon\omega)^p \wedge u| \end{aligned}$$

Applying Proposition 2.12 the last term equals

$$\sum_i \lim_{\epsilon \downarrow 0} \sup_T \int_{U_{\delta}} |(T_{ac} + \epsilon\omega)^p \wedge u| \leq \frac{C}{\text{Vol}(U_{\delta})} \sum_i \lim_{\epsilon \downarrow 0} \sup_T \int_{U_{\delta}} |(T_{ac} + \epsilon\omega)^p \wedge u|$$

which tends to 0 for $\delta \rightarrow 0$ by Prop. 2.11. □

Applying Prop. 2.13 and Prop. 2.7 in each step of the inductive construction of $\mathcal{F} \cup \mathcal{G}$ we finish the proof of the Key Lemma 2.5. □

The Key Lemma allows to construct a maximal foliation numerically trivial w.r.t. α which will be called *the* numerically trivial foliation w.r.t. α . We are now able to prove Thm. 0.7 from the introduction:

Theorem 2.14. *Let X be an n -dimensional compact Kähler manifold with Kähler form ω and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudoeffective $(1,1)$ -class with numerical dimension $\nu(\alpha)$. Let \mathcal{F} be the numerically trivial foliation w.r.t. α . Then*

$$\text{rk}(\mathcal{F}) \leq n - \nu(\alpha).$$

Proof. Set $k := \text{rk}(\mathcal{F})$. If $n - k < \alpha$ every $(n - \nu(\alpha), n - \nu(\alpha))$ -form with compact support in $X - \text{Sing}(\mathcal{F})$ is a test form for \mathcal{F} . In particular, for an arbitrarily small compact subset $K \supset \text{Sing}(\mathcal{F})$ we have

$$\lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_{X-K} (T_{ac} + \epsilon\omega)^{\nu(\alpha)} \wedge \omega^{n-\nu(\alpha)} = 0.$$

By Thm. 2.6

$$\lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_K (T_{ac} + \epsilon\omega)^{\nu(\alpha)} \wedge \omega^{n-\nu(\alpha)} \rightarrow 0$$

uniformly with the volume of K . Consequently

$$\lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_X (T_{ac} + \epsilon\omega)^{\nu(\alpha)} \wedge \omega^{n-\nu(\alpha)} = 0.$$

That contradicts the definition of $\nu(\alpha)$. \square

3. THE TRANSVERSALITY LEMMA

It is difficult to determine the numerically trivial foliation of a pseudo-effective $(1, 1)$ -class. Sometimes the following theorem helps:

Theorem 3.1 (Transversality Lemma). *Let X be an n -dimensional compact Kähler manifold with Kähler form ω , and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudo-effective $(1, 1)$ -class with numerical dimension $\nu(\alpha)$.*

Let \mathcal{F} be a foliation of rank $k = n - \nu(\alpha)$ and $\{U\}$ a covering of $X - \text{Sing}(\mathcal{F})$ by open subsets U with coordinates z_1, \dots, z_n such that the leaves of \mathcal{F} are the fibers of the projection on z_{k+1}, \dots, z_n .

Now suppose that for all $\epsilon > 0$ and all open subsets U of the covering there exists a constant $\delta_U > 0$ and a current $T_{\epsilon,U} \in \alpha[-\epsilon\omega]$ such that

$$(T_{\epsilon,U} + \epsilon\omega)|_U \geq \delta_U \cdot \omega_{\mathcal{F},U} := \delta_U \cdot \sum_{j=k+1}^n dz_j \wedge d\bar{z}_j.$$

Then \mathcal{F} is the numerically trivial foliation of α .

Proof. By Thm. 2.3 it is enough to show on every U of the covering that

$$(*) \quad \lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_U |(T_{ac} + \epsilon\omega)^p \wedge dz_I \wedge d\bar{z}_J| = 0$$

for every $1 \leq p \leq n - 1$ and every constant $(n - p, n - p)$ test form $dz_I \wedge d\bar{z}_J$ for \mathcal{F} on U . We start with proving $(*)$ for test forms $dz_I \wedge d\bar{z}_I$ with

$$m := |I \cap \{k + 1, \dots, n\}| > n - k - p.$$

From $n - k = \nu(\alpha)$ and the definition of the numerical dimension we conclude that

$$\lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_X (T_{ac} + \epsilon\omega)^p \wedge (T_{\epsilon,U,ac} + \epsilon\omega)^m \wedge \omega^{n-p-m} = 0.$$

The inequality $(T_{\epsilon,U} + \epsilon\omega)|_U \geq \delta_U \cdot \omega_{\mathcal{F},U}$ implies

$$\lim_{\epsilon \downarrow 0} \sup_{T \in \alpha[-\epsilon\omega]} \int_X (T_{ac} + \epsilon\omega)^p \wedge \omega_{\mathcal{F},U}^m \wedge \omega^{n-p-m} = 0.$$

But there exists a constant $C > 0$ such that

$$\omega_{\mathcal{F},U}^m \wedge \omega^{n-p-m} > \pm C \cdot i^{n-p} dz_I \wedge d\bar{z}_I$$

hence $(*)$ for $dz_I \wedge d\bar{z}_I$.

Next we look at general test forms $dz_I \wedge d\bar{z}_J$. We can assume w.l.o.g. that

$$|I \cap \{k + 1, \dots, n\}| > n - k - p, \quad |J \cap \{k + 1, \dots, n\}| \geq n - k - p.$$

Since

$$(T_{ac} + \epsilon\omega)^p = \sum_{|I'|=|J'|=p} T_{I'J'} dz_{I'} \wedge d\bar{z}_{J'}$$

is a smooth semipositive form outside an analytic subset we have

$$|T_{I'J'}| \leq |T_{I'I'}|^{\frac{1}{2}} \cdot |T_{J'J'}|^{\frac{1}{2}}$$

almost everywhere by the Cauchy-Schwarz inequality. Hence for $I' = \{1, \dots, n\} - I$, $J' = \{1, \dots, n\} - J$ we get

$$\begin{aligned} \int_U |(T_{ac} + \epsilon\omega)^p \wedge dz_I \wedge d\bar{z}_J| &= \int_U |T_{I'J'}| |idz_1 \wedge d\bar{z}_1 \wedge \dots \wedge idz_n \wedge d\bar{z}_n| \\ &\leq \left(\int_U |T_{I'I'}| dV \right)^{\frac{1}{2}} \cdot \left(\int_U |T_{J'J'}| dV \right)^{\frac{1}{2}} \\ &= \left(\int_U |(T_{ac} + \epsilon\omega)^p \wedge dz_I \wedge d\bar{z}_I| \right)^{\frac{1}{2}} \cdot \left(\int_U |(T_{ac} + \epsilon\omega)^p \wedge dz_J \wedge d\bar{z}_J| \right)^{\frac{1}{2}} \\ &\leq \left(\int_U |(T_{ac} + \epsilon\omega)^p \wedge dz_I \wedge d\bar{z}_I| \right)^{\frac{1}{2}} \cdot \left(\int_U (T_{ac} + \epsilon\omega)^p \wedge \omega^{n-p} \right)^{\frac{1}{2}} \end{aligned}$$

where the first inequality is a consequence of the Hölder inequality. By Boucksom's uniform estimates the second integral is uniformly bounded from above and the first factor tends to 0 when $\epsilon \rightarrow 0$ by what we have shown before.

Finally we must exclude the possibility that some foliation $\mathcal{F}' \supset \mathcal{F}$ different from \mathcal{F} is numerically trivial w.r.t. α . So let $\text{rk}(\mathcal{F}') =: k' > k$ and choose an open subset $U \subset X - (\text{Sing}(\mathcal{F}) \cup \text{Sing}(\mathcal{F}'))$ with coordinates z_1, \dots, z_n such that the leaves of \mathcal{F}' are the fibers of the projection on $z_{k'+1}, \dots, z_n$ and the leaves of \mathcal{F} the fibers of the projection on z_{k+1}, \dots, z_n . This is possible because of Prop. 1.2.

Now consider the $(n-1, n-1)$ form $\eta = \pm i^{n-1} dz_I \wedge d\bar{z}_I$ given by

$$I = \{1, \dots, k' - 1, k' + 1, \dots, n\}.$$

η is a test form for \mathcal{F}' since $|I \cap \{k' + 1, \dots, n\}| = n - k' > n - k' - 1$. But there exists a constant $C > 0$ such that

$$\int_U |(T_{ac} + \epsilon\omega) \wedge \eta| \geq \delta_u \int_U |\omega_{\mathcal{F}, U} \wedge \eta| \geq C \cdot \int_U \omega^n > 0.$$

Hence \mathcal{F}' is not numerically trivial w.r.t. α . □

As a first application we use that the foliations constructed on the surface examples 4.1 and 4.2 in [Eck04a] satisfy the conditions of the Transversality Lemma and conclude that they are still the numerically trivial foliations.

More general the definition of numerical triviality in [Eck04a] implies Def. 2.1 on surfaces.

The Transversality Lemma gives also a very simple proof for

Theorem 3.2. *Let X be a Kähler manifold and L a pseudo-effective line bundle on X . Suppose that the Kodaira-Iitaka dimension $\kappa(X, L)$ equals the numerical dimension $\nu(X, L)$ of L . Then the numerical trivial foliation of L is the Kodaira-Iitaka fibration of L .*

Proof. Choose $m \gg 0$ such that the linear system $|mL|$ defines the Kodaira-Iitaka fibration $f : X \dashrightarrow Y \subset \mathbb{P}^N$. Then the positive curvature current of the singular metric $h_{|mL|}$ satisfies the conditions in the Transversality Lemma because it is the pull back of the Fubini-Study metric on $\mathcal{O}(1)$ over Y . \square

4. VARIANTS AND DERIVED CONSTRUCTIONS

Obviously numerical triviality may also be defined with T running through subsets $\mathcal{C}(\epsilon) \subset \alpha[-\epsilon\omega]$ of $(1,1)$ -currents with analytic singularities such that

$$\mathcal{C}(\epsilon') \subset \mathcal{C}(\epsilon)$$

for $0 < \epsilon' \leq \epsilon$. The uniform boundedness of the mass of T_{ac}^p for arbitrary currents $T \in \alpha[-\epsilon\omega]$ ([Bou02, Thm. 3.1.10]) shows that we can even use a fixed positive current $T \in \alpha[0]$ with *arbitrary* singularities. In this case $(NT)_u$ reduces to

$$\int_X |T_{ac}^p \wedge u| = 0$$

for all test forms u of a foliation \mathcal{F} and Lemma 2.4 implies that locally T_{ac} is the pull back of a current S on the base of the projection locally defining \mathcal{F} . In particular this numerically trivial foliation w.r.t. T equals the one defined in [Eck04a, Def.2.9]. Furthermore let $(T_k)_{k \in \mathbb{N}}$ be a sequence of currents $T_k \in \alpha[-\epsilon_k\omega]$ with analytic singularities such that $\epsilon_k \xrightarrow{k \rightarrow \infty} 0$ and $T_{k,ac} \rightarrow T_{ac}$ almost everywhere. The Fatou lemma shows that

$$\int_X |T_{ac}^p \wedge u| \leq \liminf_{k \rightarrow \infty} \int_X |(T_{k,ac} + \epsilon_k\omega)^p \wedge u|$$

hence the numerically trivial foliation w.r.t. $(T_k)_{k \in \mathbb{N}}$ is contained in the numerically trivial foliation w.r.t. T . The same is true for the numerically trivial foliation of α . The inclusion can be strict as the surface example [Eck04a, 4.1] shows: The smooth currents $T_k \in c_1(L)[-\epsilon_k\omega]$ constructed there define a numerically trivial foliation with 1-dimensional leaves. By weak compactness of almost positive $(1,1)$ -currents there exists a subsequence of the T_k weakly converging to a positive $(1,1)$ -current in $c_1(L)$. But the only positive current in $c_1(L)$ is the integration current of a divisor whose absolutely continuous part is 0.

Considering the Iitaka fibration of a line bundle L with Kodaira dimension $\kappa(L) \geq 0$ it was proven in [Eck04a, 2.4] that this fibration is the numerically trivial foliation w.r.t. the curvature current $\Theta_{|mL|}$ of the positive metric h_{mL} defined by all sections of $|mL|$ (for an appropriate $m \gg 0$). Hence the numerically trivial foliation of $c_1(L)$ is contained in the Iitaka fibration of L .

In general it is not true that numerically trivial foliations are (rational) fibrations, see the surface examples in [Eck04a]. This motivates the following

Definition 4.1. *Let X be a compact Kähler manifold and $\alpha \in H^{1,1}(X, \mathbb{R})$ a pseudo-effective $(1,1)$ -class. Let \mathcal{F} be the numerically trivial foliation of α . Then the maximal meromorphic map $f : X \dashrightarrow Y$ such that the induced foliation is contained in \mathcal{F} is called the **pseudo-effective fibration** of α .*

Note that

- (1) Prop. 1.5 shows that the definition makes sense: There is a maximal fibration contained in a foliation.

- (2) The same definition for numerically trivial foliations w.r.t. a single positive current leads to Tsuji's numerically trivial fibrations (see [Eck04b]) by [Eck04a, Prop.2.12].

In the projective case we have an algebraic characterization of the pseudo-effective fibration. It uses Boucksom's construction of a divisorial Zariski decomposition ([Bou02, Ch.2]): For every pseudo-effective line bundle L there is a real $(1, 1)$ -class $Z(L)$ nef in codimension 1 and a negative part $N(L) = \sum_E \nu(L, E)E$ where $\nu(L, E) \neq 0$ for only finitely many exceptional divisors $E \in X$ such that

- (i) $c_1(L) = Z(L) + N(L)$ and
- (ii) the natural map $H^0(X, [kZ(L)]) \rightarrow H^0(X, kL)$ is an isomorphism for all $k \in \mathbb{N}$.

Proposition 4.2. *Let X be a complex projective manifold and L a pseudo-effective line bundle. Then there exists a covering family $(C_t)_{t \in T}$ of curves in X such that*

- (i) $C_t.(L - N(L)) = 0$ and
- (ii) the pseudo-effective fibration f of L resp. $\alpha := c_1(L)$ is the generic reduction map w.r.t. $(C_t)_{t \in T}$

Furthermore for every irreducible curve through a general point $x \in X$ not lying in a fiber of f we have

$$L.C > 0.$$

Proof. Since foliations are uniquely determined by their restriction to Zariski-open subsets (just saturate) and numerical triviality can be checked outside analytic subsets (by Theorem 2.6) numerically trivial foliations behave well under certain types of holomorphic maps: Let X be a compact Kähler manifold and $\alpha \in H^{1,1}(X, \mathbb{R})$ a pseudo-effective $(1, 1)$ -class.

- (a) Let $\pi : \widehat{X} \rightarrow X$ be a modification of compact Kähler manifolds. Then the numerically trivial foliation of $\pi^*\alpha$ on \widehat{X} is the pull back of the numerically trivial foliation of α on X , and vice versa.
- (b) The same is true for branched coverings $\pi : Y \rightarrow X$ because outside the branching locus Y may be covered by analytically open subsets biholomorphic to open subsets on X .

Finally, subfoliations \mathcal{G} of the numerically trivial foliation \mathcal{F} of α are always numerically trivial since test forms for \mathcal{G} whose support is not intersecting $\text{Sing } \mathcal{F}$ are also test forms for \mathcal{F} .

We use these observations to replace X by a desingularization $\pi : \widehat{X} \rightarrow X$ of the indeterminacy locus of the pseudo-effective fibration and L by π^*L . Let A be an ample divisor on X and $k \gg 0$. Then the curves of type

$$D_1 \cap \dots \cap D_{n-2} \cap F, \quad D_i \in |kA|,$$

where F is a fiber of the pseudo-effective fibration and $n = \dim X$, form a family (C_t) which is generically connecting on every fiber F .

There exists a composition $\pi : \widehat{X} \rightarrow X$ of modifications and a finite covering such that the strict transforms $(\overline{C_s})_{s \in S}$ of a subfamily of (C_t) are the fibers of an everywhere defined holomorphic map $f : \widehat{X} \rightarrow S$ whose induced foliation is contained in the numerically trivial foliation \widehat{F} of π^*L .

The general fiber curve $\overline{C_s}$ of f is smooth and does not intersect $\text{Sing } \mathcal{F}$ because the codimension of $\text{Sing } (\widehat{\mathcal{F}})$ is ≥ 2 . Hence there is an analytically open set $V \subset S$ such that all fiber curves over V are smooth and do not intersect $\text{Sing } (\widehat{\mathcal{F}})$.

Now we choose an $(n-1, n-1)$ test form u for $\widehat{\mathcal{F}}$ on $U := f^{-1}(V) \subset \widehat{X}$ on which we can apply Fubini's theorem:

$$\begin{aligned}
0 &= \lim_{\epsilon \downarrow 0} \sup_{T \in \pi^* \alpha[-\epsilon \widehat{\omega}]} \int_U (T_{ac} + \epsilon \widehat{\omega}) \wedge u = \lim_{\epsilon \downarrow 0} \sup_T \int_V \left(\int_{\overline{C_s}} T_{ac} + \epsilon \widehat{\omega} \right) \omega_V^{n-1} \\
&= \lim_{\epsilon \downarrow 0} \sup_T \int_V \left(\int_{\overline{C_s}} T - \sum_i \nu(T, E_i) [E_i] + \epsilon \widehat{\omega} \right) \omega_V^{n-1} \\
&= \deg \pi \cdot \lim_{\epsilon \downarrow 0} \sup_T \int_V \left(L \cdot C_t - \sum_i \nu(T, E_i) E_i \cdot C_t \right) \omega_V^{n-1} \\
&\stackrel{(*)}{=} \text{Vol}(V) \cdot (L \cdot C_t) - \lim_{\epsilon \downarrow 0} \sum_i \inf_T \nu(T, E_i) (E_i \cdot C_t) = (L - N(L)) \cdot C_t.
\end{aligned}$$

Here $\widehat{\omega}$ is a Kähler form on \widehat{X} and ω_V a Kähler form on V . The equality $(*)$ is true because there exists a sequence of currents $T_k \in \alpha[-\epsilon_k \omega]$, $\epsilon_k \xrightarrow{k \rightarrow \infty} 0$, such that

$$\nu(T_k, E) \rightarrow \nu(\alpha, E)$$

for all divisors E .

For the last claim suppose that for general points $x \in X$ there exist curves C_x with $C_x \cdot L = 0$ and C_x is not contained in any fiber of f . Using the Chow variety as in [BCE⁺00, 2.1.2] we conclude that there is a covering family $(C_t)_{t \in T}$ such that

$$C_t \cdot L = 0 \text{ and } C_t \notin \{\text{fiber of } f\}.$$

There exists a composition $\pi : \widehat{X} \rightarrow X$ of modifications and a finite covering such that the strict transforms $(\overline{C_s})_{s \in S}$ of a subfamily of (C_t) are the fibers of an everywhere defined holomorphic map $g : \widehat{X} \rightarrow S$, and still

$$\overline{C_s} \cdot \pi^* L = 0.$$

By the next proposition g is numerically trivial hence by using the Key Lemma 2.5 we see that $\pi \circ f$ is not the numerically trivial foliation of $\pi^* L$ contradicting the observations (a) and (b) from above. \square

Proposition 4.3. *Let $f : X \rightarrow S$ be a fibration of curves C_s , $s \in S$, and L a pseudo-effective line bundle on X . If $C_s \cdot L = 0$ the foliation induced by f is numerically trivial w.r.t. L .*

Proof. Remind how we proved the Transversality Lemma 3.1 by using the Cauchy-Schwarz and Hölder inequalities. This technique shows that is enough to check the numerical triviality condition $(NT)_u$ on those constant test forms u which allow us to apply Fubini's theorem as in the proof before. The proposition follows from

$$L \cdot C_t \geq (L - N(L)) \cdot C_t.$$

\square

Remark 4.4. The proof of Prop. 4.2 shows that every covering family of curves (C_t) such that the C_t 's lie inside the fibers of the pseudo-effective fibration satisfies

$$C_t \cdot (L - N(L)).$$

This is not true for the partial nef reduction defined in [BDPP04, §8]. Nevertheless this reduction map is closely related to the pseudo-effective fibration. There are two differences: First the authors use as numerical condition

$$L.C_t = 0$$

for the covering family C_t defining the reduction map. We changed this condition to

$$(L - N(L)).C_t = 0$$

because we were not able to construct a defining family (C_t) whose general member does not intersect the exceptional divisors in $N(L)$. Morally there should be such a family since exceptional divisors tend to be contractible.

Second the authors used Campana's reduction map instead of the generic reduction map. To get the same results as [BDPP04] about the Kodaira dimension of L we have to apply Campana's reduction map on the fibers of the pseudo-effective fibration of L and use the properties of Boucksom's divisorial Zariski decomposition. In the nef case all these differences vanish:

Proposition 4.5. *If L is nef the pseudo-effective fibration is the nef fibration. In particular the generic quotient of curves C with $C.L = 0$ equals the Campana quotient.*

Proof. For every curve C in a fiber of the pseudo-effective fibration there is a covering family of curves such that C is the component of one of these curves. Nefness implies $L.C = 0$. The equality of generic quotient and Campana quotient follows from [BCE⁺00, Thm. 2.4] which is of Key Lemma type. \square

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