III. Function Extensionality

Two functions should be (propositionally) equal iff their evaluation at all the arguments are (propositionally) equal.

$\mathrm{happly}: \prod(f,g : A \to B).\ f = g \to \prod(a:A).\ f(a) = g(a)$:≡ Assume f, g, p. Do induction on p. Take $\lambda(a:A).\ \mathrm{idpath}\ f(a)$.

Function Extensionality Axiom: For all $f, g : A \to B$, $\mathrm{happly}(f,g)$ is an equivalence. Call the inverse $\mathrm{funext}(f,g)$.

Remarks: (1) Function Extensionality is also an axiom in ZF set theory, in the disguise of subset extensionality: Two subsets are equal iff they contain the same elements.

(2) Function extensionality is the only way to prove equality of functions whose function rules are not judgmentally equal. $n \mapsto n + 0 \equiv n \mapsto n$, but $n \mapsto 0 + n \not\equiv n \mapsto n$ (if $+$ is defined on $\mathbb{N}$ as in I.3).

Theorem 1: Univalence $\Rightarrow$ Function Extensionality

Proof: For all functions $f, g : A \to B$ we need to show that

$$\lambda(p : f = g).\ (\lambda(a:A).\ p\ a) : f = g \to \prod(a:A).\ f(a) = g(a) \equiv f \sim g$$

is an equivalence.

(1) $A : \mathcal{U}$, $B, C : A \to \mathcal{U}$. Then: $\sum_{a:A} Ba \simeq \sum_{a:A} Ca \to \prod_{a:A} Ba \simeq Ca$.

Proof: Exercise

Apply (1) with $A :\equiv A \to B$, $B :\equiv g \mapsto f = g$, $C :\equiv g \mapsto f \sim g$ $\Rightarrow$ Enough to show $\sum_{g:A\to B} f = g \simeq \sum_{g:A\to B} f \sim g$.

(2) $\sum_{g:A\to B} f = g$ is contractible: Use $\mathrm{base} :\equiv \langle f, \mathrm{idpath}\ f\rangle$, do induction in $\sum_{g:A\to B} f = g$:

For $\langle g, p\rangle$, take $\langle p, p_*(\mathrm{idpath}\ f)\rangle : \langle f, \mathrm{idpath}\ f\rangle = \langle g, p\rangle$

$\Rightarrow$ By II.4, Lem. 1, it is enough to show: $\sum_{g:A\to B} f \sim g$ is contractible.

(3) $\sum_{g:A\to B} f \sim g$ can be retracted to $\prod_{a:A} \sum_{b:B} f(a) = b$:

$F : \langle g, H : f \sim g\rangle \mapsto (a \mapsto \langle g(a), Ha\rangle)$ $\rightsquigarrow$ $G(F(\langle g, H\rangle)) \equiv G(a \mapsto \langle g(a), Ha\rangle) \equiv \langle a \mapsto g(a), a \mapsto Ha\rangle$

$G : (a \mapsto \langle b, p : f(a) = b\rangle) \mapsto \langle a \mapsto b, a \mapsto p\rangle$ $\qquad \equiv \langle g, H\rangle$

$\Rightarrow$ By II.4, Lem. 2, it is enough to show: $\prod_{a:A} \sum_{b:B} f(a) = b$ is contractible.

Consequence of $\sum_{b:B} f(a) = b$ contractible (to $\langle f(a), \mathrm{idpath}\ f(a)\rangle$) and

Weak Function Extensionality: If $P : A \to \mathcal{U}$ is a family of contractible types then $\prod_{a:A} P(a)$ is contractible.

Theorem 2: Univalence $\Rightarrow$ Weak Function Extensionality.

Proof: Assume Univalence.

(1) If $e : A \to B$ is an equivalence, then $f \mapsto e \circ f : (X \to A) \to (X \to B)$ is an equivalence.

Univalence $\Rightarrow$ $e = \mathrm{idtoequiv}(p)$ for some $p : A = B$. Do induction on p:

For $p \equiv \mathrm{idpath}\ A$, $e \equiv \mathrm{idmap}_A$. Take the identity equivalence.

(2) Projection $pr_1 : \sum_{a:A} P(a) \to A$ is an equivalence.

Proof: Exercise (inverse is $a \mapsto \langle a, P(a)_0 \rangle$).

(1) & (2) $\Rightarrow$ $\alpha : (A \to \sum_{a:A} P(a)) \xrightarrow[pr_1 \circ -]{} (A \to A)$ is an equivalence.

II.4, Prop. 1 $\Rightarrow$ fiber $\alpha^{-1}(\mathrm{idmap}_A)$ is contractible.

(3) $\prod_{a:A} P(a)$ can be retracted to $\alpha^{-1}(\mathrm{idmap}_A) \equiv \sum (g : A \to \sum_{a:A} P(a)), pr_1 \circ g = \mathrm{idmap}_A$:

$$\varphi : f \mapsto \langle \underbrace{a \mapsto \langle a, f(a) \rangle}_{g},\ \mathrm{idpath}\ \mathrm{idmap}_A \rangle \quad \Leftarrow\ pr_1 \circ g \equiv \mathrm{idmap}_A\ !$$

$$\psi : \langle g : A \to \sum_{a:A} P(a),\ p : pr_1 \circ g = id_A \rangle \mapsto (a \mapsto \underbrace{p(a)_* \, \underbrace{pr_2(g(a))}_{P(pr_1 \circ g(a))}}_{: P(a)})$$

$$\psi(\varphi(f)) \equiv a \mapsto f(a) \equiv f$$

II.4, Lem. 2 & (3) $\Rightarrow$ $\prod_{a:A} P(a)$ is contractible.